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Gauge theory on fuzzy $S^2 \times S^2$ and regularization on noncommutative $\mathbb{R}^4$

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Abstract: We define U(n) gauge theory on fuzzy $S^2_N \times S^2_N$ as a multi-matrix model, which reduces to ordinary Yang-Mills theory on $S^2 \times S^2$ in the commutative limit $N \to \infty$. The model can be used as a regularization of gauge theory on noncommutative $\mathbb{R}^4_\theta$ in a particular scaling limit, which is studied in detail. We also find topologically non-trivial U(1) solutions, which reduce to the known "fluxon" solutions in the limit of $\mathbb{R}^4_\theta$, reproducing their full moduli space. Other solutions which can be interpreted as 2-dimensional branes are also found. The quantization of the model is defined non-perturbatively in terms of a path integral which is finite. A gauge-fixed BRST-invariant action is given as well. Fermions in the fundamental representation of the gauge group are included using a formulation based on SO(6), by defining a fuzzy Dirac operator which reduces to the standard Dirac operator on $S^2 \times S^2$ in the commutative limit. The chirality operator and Weyl spinors are also introduced.

Keywords: Nonperturbative Effects, Solitons Monopoles and Instantons, Non-Commutative Geometry, Matrix Models.

Contents

1. Introduction

Gauge theories on noncommutative spaces have received much attention in recent years. One of the reasons is the natural realization of such theories in the framework of string theory and D-branes [1], however they deserve interest also in their own right; see [2, 3] for some reviews. One of the most remarkable new features of noncommutative gauge theories is the fact that they can be defined in terms of multi-matrix models, which means that the action involves only products of "covariant coordinates" $X_i = x_i + A_i$, with gauge transformations acting as $X_i \to UX_iU^{-1}$. In particular for certain quantized compact spaces such as fuzzy spheres and tori, these X_i are finite-dimensional hermitean matrices of size N. Nevertheless, the conventional gauge theory is correctly reproduced in the limit $N \to \infty$. This leads to a natural quantization prescription by simply integrating over these matrices. For the much-studied case of the quantum plane $\mathbb{R}^d_\theta$, the matrices X_i are infinite-dimensional, and the precise definition of the models is quite non-trivial. This is particularly obvious by noting that the naive action for gauge theory on $\mathbb{R}^d_\theta$ contains sectors with any rank of the gauge group U(n) [4]. To have a well-defined theory and quantization prescription, a regularization of gauge theory on $\mathbb{R}^d_\theta$ based on the finite compact case is therefore very desirable. Furthermore, the formulation as multi-matrix model leads to the hope that non-trivial results may be obtained using the sophisticated techniques from random matrix theory. We introduce in this paper such a matrix model for fuzzy $S^2 \times S^2$, and study its relationship with $\mathbb{R}^4_\theta$.

In the 2-dimensional case, this matrix-model approach to gauge theory has been studied in considerable detail for the fuzzy sphere S^2_N [5–10] and the noncommutative torus $\mathbb{T}^2_\theta$ [11–14], both on the classical and quantized level. It is well-known that $\mathbb{R}^2_\theta$ can be obtained as scaling limits of these spaces S^2_N and $\mathbb{T}^2_N$ at least locally, which suggests a correspondence also for the gauge theories. This correspondence of gauge theories has been studied in great detail for the case of $\mathbb{T}^2_\theta \to \mathbb{R}^2_\theta$ [12, 15, 16] on the quantized level, exhibiting the role of certain instanton contributions. A matching of gauge theory on the classical level can also be seen for $S^2_N \to \mathbb{R}^2_\theta$ [17, 18], which is implicitly contained in section 7 of the present paper.

In 4 dimensions, the quantization of gauge theory is more difficult, and a regularization using finite-dimensional matrix models is particularly important. The most obvious 4-dimensional spaces suitable for this purpose are $\mathbb{T}^4, S^2 \times S^2$ and $\mathbb{C}P^2$. On fuzzy $\mathbb{C}P^2_N$ [19–21], such a formulation of gauge theory was given in [22]. This can indeed be used to obtain $\mathbb{R}^4_\theta$ for the case of U(2) -invariant θ_{ij}. The case of $\mathbb{R}^2 \times S^2_N$ as regularization of $\mathbb{R}^4_\theta$

with degenerate θ_{ij} was considered in [18, 23], exhibiting a relation with a conventional non-linear sigma model. A formulation of lattice gauge theory for even-dimensional tori has been discussed in [24, 25, 14]. Related “fuzzy” solutions of the string-theoretical matrix models [26] were studied e.g. in [27, 28], see also [29].

In the present paper we give a definition of $\mathrm{U}(n)$ gauge theory on fuzzy $S^2_N \times S^2_N$, which can be used to obtain any $\mathbb{R}^4_\theta$ as a scaling limit. The action is a simple generalization of the matrix model approach of [7] for fuzzy S^2_N. It differs from similar string-theoretical matrix models [26] by adding a constraint-term, which ensures that the “vacuum” solution is stable and describes the product of 2 spheres. The fluctuations of the covariant coordinates then correspond as usual to the gauge fields, and the action reduces to ordinary Yang-Mills theory on $S^2 \times S^2$ in the limit $N \to \infty$. The quantization of the model is defined by a finite integral over the matrix degrees of freedom, which is shown to be convergent due to the constraint term. We also give a gauge-fixed action with BRST symmetry.

We then discuss some features of the model, in particular a hidden SO(6) invariance of the action which is broken explicitly by the constraint. This suggests some alternative formulations in terms of “collective matrices”, which are assembled from the individual covariant coordinates (matrices). This turns out to be very useful to construct a Dirac operator, and may help to eventually study the quantization of the model. The stability of the model without constraint is also discussed, and we show that the only flat directions of the SO(6) -invariant action are fluctuations of the constant radial modes of the 2 spheres.

As a further test of the proposed gauge theory, we study in section 6 topologically non-trivial solutions (instantons) on $S^2_N \times S^2_N$. We find in particular a simple class of solutions which can be interpreted as U(1) instantons with quantized flux, combined with a singular, localized “flux tube”. They are related to the so-called “fluxon” solutions of U(1) gauge theory on $\mathbb{R}^4_\theta$. Solutions which can be interpreted as 2-dimensional spherical branes wrapping one of the two spheres are also found and are matched with similar solutions on $\mathbb{R}^4_\theta$. We then study the relation of the model on $S^2_N \times S^2_N$ with Yang-Mills theory on $\mathbb{R}^4_\theta$, and demonstrate that the usual Yang-Mills action on $\mathbb{R}^4_\theta$ is recovered in the appropriate scaling limit. Some aspects of U(1) instantons (“fluxons”) on $\mathbb{R}^4_\theta$ are recalled in section 7.2, and we show in detail how they arise as limits of the above non-trivial solutions on $S^2_N \times S^2_N$. In particular, we are able to match the moduli space of n fluxons, corresponding to their location on $\mathbb{R}^4_\theta$ resp. $S^2_N \times S^2_N$. We find in particular that even though the field strength in the “bulk” vanishes in the limit of $\mathbb{R}^4_\theta$, it does contribute to the action on $S^2_N \times S^2_N$ with equal weight as the localized flux tube. This can be interpreted on $\mathbb{R}^4_\theta$ as a topological or surface term at infinity. Another unexpected feature on $S^2_N \times S^2_N$ is the appearance of certain “superselection rules”, restricting the possible instanton numbers. In other words, not all instanton numbers on $\mathbb{R}^4_\theta$ are reproduced for a given matrix size $\mathcal{N}$, however they can be found by considering matrices of different size. This depends on the precise form of the constraint term in the action, which is hence seen to imply also certain topological constraints. To recover the full space of ADHM solutions on $\mathbb{R}^4_\theta$ starting from $S^2_N \times S^2_N$ remains an open challenge, which is non-trivial since the concept of self-duality does not extend naturally to the fuzzy case.

We should mention here that topologically non-trivial configurations have also been discussed more abstractly in terms of projective modules using a somewhat different formulation of gauge theory on fuzzy spaces, see in particular [30, 31].

Finally in section 8 we include charged fermions in the fundamental representation of the gauge group, by giving a Dirac operator $\widehat{D}$ which in the large-N limit reduces to the ordinary gauged Dirac operator on $S^2 \times S^2$. This Dirac operator is covariant under the SO(6) symmetry of the embedding space $S^2 \times S^2 \subset \mathbb{R}^6$, and exactly anti-commutes with a chirality operator. The 4-dimensional physical Dirac spinors are obtained by suitable projections from 8-dimensional SO(6) spinors. This projection however commutes with $\widehat{D}$ only in the large-N limit, and is achieved by giving one of the 2 spinors a large mass. Weyl spinors can then be defined using the exact chirality operator.

2. The fuzzy spaces S^2_N and $S^2_{N_L} \times S^2_{N_R}$

We start by recalling the definition of the fuzzy sphere in order to fix our conventions and notation. The algebra of functions on the fuzzy sphere is the finite algebra S^2_N generated by hermitean operators $x_i = (x_1, x_2, x_3)$ satisfying the defining relations

$$[x_i, x_j] = i\Lambda_N \epsilon_{ijk} x_k \,, \tag{2.1}$$

$$x_1^2 + x_2^2 + x_3^2 = R^2 \,. \tag{2.2}$$

They are obtained from the N-dimensional representation of $su(2)$ with generators λ_i ($i = 1, 2, 3$) and commutation relations

$$[\lambda_i, \lambda_j] = i\epsilon_{ijk}\lambda_k \,, \qquad \sum_{i=1}^{3} \lambda_i \lambda_i = \frac{N^2-1}{4} \tag{2.3}$$

(see appendix A) by identifying

$$x_i = \Lambda_N \, \lambda_i \,, \qquad \Lambda_N = \frac{2R}{\sqrt{N^2-1}} \,. \tag{2.4}$$

The noncommutativity parameter Λ_N is of dimension length. The algebra of functions S^2_N therefore coincides with the simple matrix algebra $Mat(N, \mathbb{C})$. The normalized integral of a function $f \in S^2_N$ is given by the trace

$$\int_{S^2_N} f = \frac{4\pi R^2}{N} \mathrm{tr}(f) \,. \tag{2.5}$$

The functions on the fuzzy sphere can be mapped to functions on the commutative sphere S^2 using the decomposition into harmonics under the action

$$J_i f = [\lambda_i, f] \tag{2.6}$$

of the rotation group SU(2). One obtains analogs of the spherical harmonics up to a maximal angular momentum $N-1$. Therefore S^2_N is a regularization of S^2 with a UV cutoff, and the commutative sphere S^2 is recovered in the limit $N \to \infty$. Note also that for the standard representation (A.2), entries in the upper-left block of the matrices correspond to functions localized at $x_3 = R$. In particular, the fuzzy delta-function at the "north pole" is given by a suitably normalized projector of rank 1,

$$\delta^{(2)}_{NP}(x) = \frac{N}{4\pi R^2}\left|\frac{N-1}{2}\right\rangle\left\langle\frac{N-1}{2}\right| \tag{2.7}$$

where $|\frac{N-1}{2}\rangle$ is the highest weight state with maximal eigenvalue of λ_3. Delta-functions with arbitrary localization are obtained by rotating (2.7).

The simplest 4-dimensional generalization of the above is the product $S^2_{N_L} \times S^2_{N_R}$ of 2 such fuzzy spheres, with generally independent parameters $N_{L,R}$. It is generated by a double set of representations of $su(2)$ commuting with each other, i.e. by λ^L_i, λ^R_i satisfying

$$\begin{aligned} &[\lambda^L_i, \lambda^L_j] = i\epsilon_{ijk}\lambda^L_k\,, \qquad [\lambda^R_i, \lambda^R_j] = i\epsilon_{ijk}\lambda^R_k\,,\\ &[\lambda^L_i, \lambda^R_j] = 0 \end{aligned}$$

for $i, j = 1, 2, 3$, and Casimirs

$$\sum_{i=1}^{3}\lambda^L_i\lambda^L_i = \frac{N_L^2-1}{4}\,, \qquad \sum_{i=1}^{3}\lambda^R_i\lambda^R_i = \frac{N_R^2-1}{4}\,. \tag{2.8}$$

This can be realized as a tensor product of 2 fuzzy sphere algebras

$$\lambda^L_i = \lambda_i \otimes 1_{N_R\times N_R}\,, \tag{2.9}$$

$$\lambda^R_i = 1_{N_L\times N_L} \otimes \lambda_i\,, \tag{2.10}$$

hence as algebra we have $S^2_{N_L} \times S^2_{N_R} \cong \mathrm{Mat}(\mathcal{N}, \mathbb{C})$ where

$$\mathcal{N} = N_L N_R\,. \tag{2.11}$$

The normalized coordinate functions are given by

$$x^{L,R}_i = \frac{2R}{\sqrt{(N^{L,R})^2-1}}\,\lambda^{L,R}_i\,, \qquad \sum(x^L_i)^2 = R^2 = \sum(x^R_i)^2\,. \tag{2.12}$$

This space[1] can be viewed as regularization of $S^2 \times S^2 \subset \mathbb{R}^6$, and admits the symmetry group $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \subset \mathrm{SO}(6)$. The generators $x^{L,R}_i$ should be viewed as coordinates in an embedding space $\mathbb{R}^6$. The normalized integral of a function $f \in S^2_{N_L} \times S^2_{N_R}$ is now given by

$$\int\limits_{S^2_{N_L}\times S^2_{N_R}} f = \frac{16\pi^2R^4}{\mathcal{N}}\mathrm{tr}(f) = \frac{V}{\mathcal{N}}\mathrm{tr}(f)\,, \tag{2.13}$$

where we define the volume $V := 16\pi^2 R^4$. We will mainly consider $N_L = N_R$ in the following.

[1] In principle one could also introduce different radii $R^{L,R}$ for the 2 spheres, but for simplicity we will keep only one scale parameter R (and usually we will set $R = 1$).

2.1 The quantum plane limit $\mathbb{R}^4_\theta$

It is well-known [32] that if a fuzzy sphere is blown up near a given point, it can be used to obtain a (compactified) quantum plane: Consider the tangential coordinates $x_{1,2}$ near the "north pole". Setting

$$R^2 = \frac{N\theta}{2}\,, \tag{2.14}$$

they satisfy the commutation relations

$$[x_1, x_2] = i\frac{2R}{N}x_3 = i\frac{2R}{N}\sqrt{R^2 - x_1^2 - x_2^2} = i\theta\left(1 + O\left(\frac{1}{N}\right)\right). \tag{2.15}$$

Therefore in the large-N limit with (2.14) keeping θ fixed, we recover[2] the commutation relation of the quantum plane,

$$[x_1, x_2] = i\theta \tag{2.16}$$

up to corrections of order $1/N$. Similarly, starting with $S^2_{N_L} \times S^2_{N_R}$ and setting

$$R^2 = \frac{N_{L,R}\,\theta_{L,R}}{2}\,, \tag{2.17}$$

we obtain in the large N_L, N_R limit

$$\begin{aligned} [x_i^L, x_j^L] &= i\epsilon_{ij}\theta^L\,, \qquad [x_i^R, x_j^R] = i\epsilon_{ij}\theta^R\,, \\ \left[x_i^L, x_j^R\right] &= 0\,. \end{aligned} \tag{2.18}$$

This is the most general form of $\mathbb{R}^4_\theta$ with coordinates $(x_1, \ldots, x_4) \equiv (x_1^L, x_2^L, x_1^R, x_2^R)$ (after a suitable orthogonal transformation). The integral of a function $f(x)$ then becomes

$$\int\limits_{S^2_{N_L}\times S^2_{N_R}} f(x) \to 4\pi^2\theta_L\theta_R \mathrm{tr}(f(x)) =: \int\limits_{\mathbb{R}^4_\theta} f(x)\,, \tag{2.19}$$

which has indeed the standard normalization, giving each "Planck cell" the appropriate volume.

3. Gauge theory on fuzzy $S^2 \times S^2$

We start with the most general case, and construct a matrix model having $S^2_{N_L} \times S^2_{N_R}$ as its ground state. The fluctuations around this ground state will produce a gauge theory. A simplified and more elegant formulation in terms of "collective matrices" similar as in [7] for the fuzzy sphere will be given later in section 4.

In the fuzzy case, it is natural to construct $S^2_L \times S^2_R$ as "submanifold" of $\mathbb{R}^6$. We therefore consider a multi-matrix model with 6 dynamical fields ("covariant coordinates")

[2] One could be more sophisticated and use the stereographic projections as in [32], which leads essentially to the same results.

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B_i^L and B_i^R ($i = 1,2,3$), which are $\mathcal{N} \times \mathcal{N}$ hermitean matrices. As action we choose the following generalization of the action in [7, 8],

$$S = \frac{1}{g^2} \int \frac{1}{2} F_{ia\,jb} F_{ia\,jb} + \varphi_L^2 + \varphi_R^2 \tag{3.1}$$

with $a, b = L, R$ and $i, j = 1, 2, 3$; summation over repeated indices is implied. Here $\varphi_{L,R}$ are defined as

$$\varphi_L := \frac{1}{R^2}\left(B_i^L B_i^L - \frac{N_L^2 - 1}{4}\right), \qquad \varphi_R := \frac{1}{R^2}\left(B_i^R B_i^R - \frac{N_R^2 - 1}{4}\right), \tag{3.2}$$

and the terms $\varphi_L^2 + \varphi_R^2$ in the action ensure that the unwanted radial degrees of freedom are suppressed [7, 8]. R denotes the radius of the two spheres, which we keep explicitly to have the correct dimensions. The field strength is defined by

$$\begin{aligned} F_{iL\,jL} &= \frac{1}{R^2}(i[B_i^L, B_j^L] + \epsilon_{ijk} B_k^L)\,, \\ F_{iR\,jR} &= \frac{1}{R^2}(i[B_i^R, B_j^R] + \epsilon_{ijk} B_k^R)\,, \\ F_{iL\,jR} &= \frac{1}{R^2}(i[B_i^L, B_j^R])\,. \end{aligned} \tag{3.3}$$

This model (3.1) is manifestly invariant under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ rotations acting in the obvious way, and $\mathrm{U}(\mathcal{N})$ gauge transformations acting as $B_i^{L,R} \to U B_i^{L,R} U^{-1}$. We will see below that this reduces indeed to the U(1) Yang-Mills action on $S^2 \times S^2$ in the commutative limit. Note that if the action (3.1) is considered as a matrix model, the radius drops out using (2.13). The equations of motion (e.o.m.) for B_i^L are

$$\begin{aligned} &\left\{B_i^L, B_j^L B_j^L - \frac{N_L^2 - 1}{4}\right\} + (B_i^L + i\epsilon_{ijk} B_j^L B_k^L) + \\ &+ i\epsilon_{ijk}[B_j^L, (B_k^L + i\epsilon_{krs} B_r^L B_s^L)] + [B_j^R, [B_j^R, B_i^L]] = 0\,, \end{aligned} \tag{3.4}$$

and those for B_i^R are obtained by exchanging $L \leftrightarrow R$. By construction, the minimum or ground state of the action is given by $F = \varphi = 0$, hence $B_i^{L,R} = \lambda_i^{L,R}$ as in (2.9), (2.10) up to gauge transformations; cp. [22] for a similar approach on $\mathbb{C}P^2$. We can therefore expand the "covariant coordinates" B_i^L and B_i^R around the ground state

$$B_i^a = \lambda_i^a + R A_i^a\,, \tag{3.5}$$

where $a \in \{L, R\}$ and A_i^a is small. Then $A_i^{L,R}$ transforms under gauge transformations as

$$A_i^{L,R} \to A'^{L,R}_i = U A_i^{L,R} U^{-1} + U[\lambda_i^{L,R}, U^{-1}]\,, \tag{3.6}$$

and the field strength takes a more familiar form,[3]

$$\begin{aligned} F_{iL\,jL} &= i\left(\left[\frac{\lambda_i^L}{R}, A_j^L\right] - \left[\frac{\lambda_j^L}{R}, A_i^L\right] + [A_i^L, A_j^L]\right), \\ F_{iR\,jR} &= i\left(\left[\frac{\lambda_i^R}{R}, A_j^R\right] - \left[\frac{\lambda_j^R}{R}, A_i^R\right] + [A_i^R, A_j^R]\right), \\ F_{iL\,jR} &= i\left(\left[\frac{\lambda_i^L}{R}, A_j^R\right] - \left[\frac{\lambda_j^R}{R}, A_i^L\right] + [A_i^L, A_j^R]\right). \end{aligned} \tag{3.7}$$

[3]We do not distinguish between upper and lower indices L, R.

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So far, the spheres are described in terms of 3 cartesian covariant coordinates each. In the commutative limit, we can separate the radial and tangential degrees of freedom. There are many ways to do this; perhaps the most elegant for the present purpose is to note that the terms $\int \varphi_L^2 + \varphi_R^2$ in the action imply that $\varphi_{L,R}$ is bounded for configurations with finite action. Using

$$\varphi_L = \frac{\lambda_i^L}{R} A_i^L + A_i^L \frac{\lambda_i^L}{R} + A_i^L A_i^L\,, \tag{3.8}$$

and similarly for φ_R it follows that

$$x_i A_i^a + A_i^a x_i = O(\frac{\varphi}{N}) \tag{3.9}$$

for finite A_i^a. This means that A_i^a is tangential in the (commutative) large-N limit. Alternatively, one could consider $\phi_L = N\varphi_L$, which would acquire a mass of order N and decouple from the other fields.[4] The commutative limit of (3.1) therefore gives the standard action for electrodynamics on $S^2 \times S^2$,

$$S = \frac{1}{2g^2} \int\limits_{S^2 \times S^2} F^t_{ia\,jb} F^t_{ia\,jb}$$

with $a, b = L, R$ and $i, j = 1, 2, 3$. Here $F^t_{iL\,jR}$ denotes the usual tangential field strength. This can be seen most easily noting that e.g. at the north pole $x_3^{L,R} = R$, one can replace

$$i[\frac{\lambda_i^{L,R}}{R}, \cdot] \quad \rightarrow \quad -\varepsilon_{ij} \frac{\partial}{\partial x_j^{L,R}} \tag{3.10}$$

in the commutative limit, so that upon identifying the commutative gauge fields $A_i^{(cl)}$ via

$$A_i^{(cl)L,R} = -\varepsilon_{ij} A_i^{L,R} \tag{3.11}$$

the field strength is given by the standard expression $F^t_{iL\,jR} = \partial_i^L A_j^{(cl)R} - \partial_j^R A_i^{(cl)L}$ etc.

U(k) **gauge theory.** The above action generalizes immediately to the nonabelian case, keeping precisely the same action (3.1), (3.2) but replacing the matrices $B_i^{L,R}$ by $k\mathcal{N} \times k\mathcal{N}$ matrices, cp. [7]. Expanding them in terms of (generalized) Gell-Mann matrices, the same action (3.1) is the fuzzy version of nonabelian U(k) Yang-Mills on $S^2 \times S^2$.

4. A formulation based on SO(6)

The above action can be cast into a nicer form by assembling the matrices $B_i^{L,R}$ into bigger "collective matrices", following [7]. Since it is natural from the fuzzy point of view to embed $S^2 \times S^2 \subset R^6$ with corresponding embedding of the symmetry group $SO(3)_L \times SO(3)_R \subset SO(6)$, we consider

$$B_\mu = (B_i^L, B_i^R) \tag{4.1}$$

[4]The constraints $\varphi_L = 0 = \varphi_R$ could also be imposed by hand; however the suppression through the above terms in the action is more flexible, as we will see in section 6.

(Greek indices μ, ν denoting from now on all the six dimensions) to be the 6 -dimensional irrep of $so(6) \cong su(4)$. Since $(4)\otimes(4) = (6)\oplus(10)$, it is natural to introduce the intertwiners

$$\gamma_\mu = (\gamma_i^L, \gamma_i^R) = (\gamma_\mu)^{\alpha,\beta} \tag{4.2}$$

of $(6) \subset (4) \otimes (4)$. where α, β denote indices of (4). We could then assemble our dynamical fields into a single $4\mathcal{N} \times 4\mathcal{N}$ matrix

$$B = B_\mu \gamma_\mu + \text{const} \cdot \mathbb{1}\,. \tag{4.3}$$

Of course the most general such $4\mathcal{N}\times 4\mathcal{N}$ matrix contains far too many degrees of freedom, and we have to constrain these B further. Since SU(4) acts on B as $B \to U^T BU$, the γ_μ can be chosen as totally anti-symmetric matrices, which precisely singles out the $(6) \subset (4)\otimes(4)$. One can moreover impose

$$(\gamma_i^L)^\dagger = \gamma_i^L\,, \qquad (\gamma_i^R)^\dagger = -\gamma_i^R\,, \tag{4.4}$$

and

$$\gamma_i^L \gamma_j^L = \delta_{ij} + i\epsilon_{ijk}\gamma_k^L\,, \tag{4.5}$$

$$\gamma_i^R \gamma_j^R = -\delta_{ij} - \epsilon_{ijk}\gamma_k^R\,, \tag{4.6}$$

$$[\gamma_i^L, \gamma_j^R] = 0\,, \tag{4.7}$$

which will be assumed from now on; we will give two explicit such representations in (B.5), (D.2). This would suggest to constrain B to be antisymmetric. However, the component fields B_μ are naturally considered as hermitean rather than symmetric matrices. Furthermore, since the $\gamma_\mu = (\gamma_\mu)^{\alpha,\beta}$ have two upper indices, they do not form an algebra. There are now 2 ways to proceed. We can either separate them again by introducing two $4\mathcal{N} \times 4\mathcal{N}$ matrices,

$$B^L = \frac{1}{2} + B_i^L \gamma_i^L\,, \qquad B^R = \frac{i}{2} + B_i^R \gamma_i^R\,, \tag{4.8}$$

breaking SO(6) $\to$ SO(3) $\times$ SO(3). This will be pursued in appendix B. Alternatively, we can use the γ_μ with the above properties to construct the 8×8 Gamma-matrices

$$\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^{\mu\dagger} & 0 \end{pmatrix}, \tag{4.9}$$

which generate the SO(6)-Clifford algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = \begin{pmatrix} \gamma^\mu\gamma^{\nu\dagger} + \gamma^\nu\gamma^{\mu\dagger} & 0 \\ 0 & \gamma^{\mu\dagger}\gamma^\nu + \gamma^{\nu\dagger}\gamma^\mu \end{pmatrix} = 2\delta^{\mu\nu}\,. \tag{4.10}$$

This suggests to consider the single hermitean $8\mathcal{N} \times 8\mathcal{N}$ matrix

$$C = \Gamma^\mu B_\mu + C_0 = \begin{pmatrix} 0 & B^L \\ B^L & 0 \end{pmatrix} + \begin{pmatrix} 0 & B^R \\ -B^R & 0 \end{pmatrix} =: C^L + C^R\,, \tag{4.11}$$

where $C_0 = C_0^L + C_0^R$ denote the constant 8×8-matrices

$$C_0^L = -\frac{i}{2}\Gamma_1^L\Gamma_2^L\Gamma_3^L = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.12}$$

$$C_0^R = -\frac{i}{2}\Gamma_1^R\Gamma_2^R\Gamma_3^R = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.13}$$

in the above basis. This is very close to the approach in [7], and using the Clifford algebra and the above definitions one finds indeed

$$C^2 = B_\mu B_\mu + \frac{1}{2} + \Sigma_8^{\mu\nu} F_{\mu\nu}\,. \tag{4.14}$$

Here $\Sigma_8^{\mu\nu} = -\frac{i}{4}[\Gamma_\mu, \Gamma_\nu]$, and the field strength $F_{\mu\nu}$ coincides with the definition in (3.3) if written in the $L-R$ notation,

$$F_{ia\,jb} = i[B_{ia}, B_{jb}] + \delta_{ab}\epsilon_{ijk}B_{ka}\,.$$

Therefore the action

$$S_6 = \mathrm{Tr}\left(\left(C^2 - \frac{N^2}{2}\right)^2\right) = 8\mathrm{tr}\left(B_\mu B_\mu - \frac{N^2-1}{2}\right)^2 + 4\mathrm{tr}F_{\mu\nu}F_{\mu\nu} \tag{4.15}$$

is quite close to what we want. The only difference is the term $(B_\mu B_\mu - \frac{N^2-1}{2})^2$ instead of $(B_{iL}B_{iL} - \frac{N_L^2-1}{4})^2 + (B_{iR}B_{iR} - \frac{N_R^2-1}{4})^2$, for $2N^2 = N_L^2 + N_R^2$. This difference is easy to understand: since (4.15) is SO(6)-invariant, the ground state should be some S^5. We therefore have to break this SO(6)- invariance explicitly, which will be done in the next section. However before doing that, let us try to understand action (4.15) better and see whether it leads to a meaningful 4-dimensional field theory. We show in appendix C by carefully integrating out the scalar components of $B_i^{L,R}$ that the SO(6)- invariant constraint term in (4.15) induces the second term in the following effective action

$$S_6^{\mathrm{eff}} \sim 4\mathrm{tr}\left(F_{\mu\nu}F_{\mu\nu} - (F_{iL}x_{iL} - F_{iR}x_{iR})\frac{1}{4(\frac{1}{2} - \partial_\mu\partial_\mu)}(F_{iL}x_{iL} - F_{iR}x_{iR})\right) \tag{4.16}$$

in the commutative limit, where $F_{iL} = \frac{1}{2}\epsilon_{ijk}F_{jL\,kL}$ etc. Comparing the second term with $F_{\mu\nu}F_{\mu\nu}$, we see that the zero mode of the Laplace operator $\partial_\mu\partial_\mu$ can produce a contribution that cancels the corresponding contribution from $F_{\mu\nu}F_{\mu\nu}$, but that all higher modes are smaller by at least a factor of $2(\frac{1}{2} - \partial_\mu\partial_\mu)$. Therefore, the action (4.15) is positive definite except for the obvious zero mode $\delta B_i^L = \epsilon$, $\delta B_i^R = -\epsilon$. This means that the geometry of $S_L^2 \times S_R^2$ is locally stable even with the SO(6)-symmetry unbroken, except for opposite fluctuations of the radii.

4.1 Breaking SO(6) $\rightarrow$ SO(3) $\times$ SO(3)

To obtain the original action (3.1) for $S^2 \times S^2$, we have to break the SO(6)-symmetry down to SO(3) $\times$ SO(3). We can do this by using the left and right gauge fields C^L and

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C^R introduced in (4.11) separately. Their squares are

$$C_L^2 = B_{iL}B_{iL} + \frac{1}{4} + \begin{pmatrix} \gamma_L^i & 0 \\ 0 & \gamma_L^i \end{pmatrix} (B_{iL} + i\epsilon_{ijk}B_{jL}B_{kL}) \,,$$
$$C_R^2 = B_{iR}B_{iR} + \frac{1}{4} - i \begin{pmatrix} \gamma_R^i & 0 \\ 0 & \gamma_R^i \end{pmatrix} (B_{iR} + i\epsilon_{ijk}B_{jR}B_{kR}) \,.$$

As both γ_L^i, γ_R^i and $\gamma_L^i\gamma_R^j$ are traceless, we have

$$S_{\text{break}} := \mathrm{Tr}\Big(\Big(C_L^2 - \frac{N_L^2}{4}\Big)\Big(C_R^2 - \frac{N_R^2}{4}\Big)\Big) = 8\mathrm{Tr}\Big(\Big(B_{iL}B_{iL} - \frac{N_L^2-1}{4}\Big)\Big(B_{iR}B_{iR} - \frac{N_R^2-1}{4}\Big)\Big) \,.$$

With these terms we can recover our action as

$$\begin{aligned} S &= S_6 - 2S_{\text{break}} \\ &= \mathrm{Tr}\Big(\Big(C^2 - \frac{N^2}{2}\Big)^2 - \Big\{C_L^2 - \frac{N_L^2}{4}, C_R^2 - \frac{N_R^2}{4}\Big\}\Big) \\ &= 8\,\mathrm{tr}\left(\Big(B_{iL}B_{iL} - \frac{N_L^2-1}{4}\Big)^2 + \Big(B_{iR}B_{iR} - \frac{N_R^2-1}{4}\Big)^2 + \frac{1}{2}F_{\mu\nu}F_{\mu\nu}\right), \end{aligned} \tag{4.17}$$

which is precisely the action (3.1) for gauge theory on $S^2_{N_L} \times S^2_{N_R}$ omitting the overall constants. Hence the action is formulated as 2-matrix model, however with highly constrained matrices C_L, C_R. This formulation using the Gamma-matrices is very natural and useful if one wants to couple the gauge fields to fermions, as discussed in section 8.

For simplicity, we will only consider $N_L = N_R = N$ from now on.

5. Quantization

The quantization of the gauge theory defined by (3.1) or its reformulation (4.17) is straightforward in principle, by a "path integral" over the hermitean matrices

$$Z[J] = \int dB_\mu e^{-S[B_\mu] + \mathrm{tr} B_\mu J_\mu} \,. \tag{5.1}$$

Note that there is no need to fix the gauge since the gauge group $U(\mathcal{N})$ is compact. The above path integral is well-defined and finite for any fixed $\mathcal{N}$. To see this, it is enough to show that the integral $\int dB_\mu \exp(-(B_i^L B_i^L - (N^2-1)/4)^2 - (B_i^R B_i^R - (N^2-1)/4)^2)$ converges, since the contributions from the field strength further suppress the integrand. This integral is obviously convergent for any fixed N.

For perturbative computations it is necessary to fix the gauge, and to substitute gauge invariance by BRST-invariance. Such a gauge-fixed action will be presented next.

5.1 BRST symmetry

To construct a gauge-fixed BRST-invariant action, we have to introduce ghost fields c and anti-ghost fields $\bar{c}$. These are fermionic fields, more precisely $\mathcal{N}\times\mathcal{N}-$ matrices with entries which are Grassman variables.

The full gauge-fixed action reads:

$$S_{\mathrm{BRST}} = S + \frac{1}{\mathcal{N}}\mathrm{tr}\left(\bar{c}[\lambda_\mu,[B_\mu,c]] - \left(\frac{\alpha}{2}b - [\lambda_\mu,B_\mu]\right)b\right),$$

where b is an auxiliary (Nakanishi-Lautrup) field. This action is invariant with respect to the following BRST-transformations:

$$sB_\mu = [B_\mu, c] \qquad sc = cc \tag{5.2}$$

$$s\bar{c} = b \qquad sb = 0 \tag{5.3}$$

(matrix product is understood), where the BRST-differential s acts on a product of fields as follows:

$$s(XY) = X(sY) + (-1)^{\varepsilon_Y}(sX)Y\,.$$

Here ε_Y denotes the Grassman-parity of Y

$$\varepsilon_Y = \begin{cases} 0 & Y\text{ bosonic} \\ 1 & Y\text{ fermionic}\,. \end{cases}$$

As usual, it is not difficult to check that these BRST-transformations are indeed nilpotent, i.e.

$$s^2 = 0\,.$$

Integrating out the auxiliary field b leads to the following action

$$S'_{\mathrm{BRST}} = S + \frac{1}{\mathcal{N}}\mathrm{tr}(\bar{c}[\lambda_\mu,[B_\mu,c]] - \frac{1}{2\alpha}[\lambda_\mu,B_\mu][\lambda_\nu,B_\nu])\,.$$

Setting $\alpha = 1$ corresponds to the Feynman gauge. This is indeed what one would obtain by the Faddeev-Popov procedure. The action S' is invariant with respect to the following operations:

$$\begin{aligned} s'B_\mu &= [B_\mu, c] \\ s'c &= cc \\ s'\bar{c} &= [\lambda_\mu, B_\mu]\,. \end{aligned}$$

Since we have used the equations of motion of b, the BRST-differential s' is *not* nilpotent off-shell anymore but still we have

$$s'^2|_{\mathrm{on-shell}} = 0\,.$$

6. Topologically non-trivial solutions on $S^2_N \times S^2_N$

In order to understand better the non-trivial solutions found below, we first note that the classical space $S^2 \times S^2$ is symplectic with symplectic form

$$\omega = \omega^L + \omega^R\,, \tag{6.1}$$

where

$$\omega^L = \frac{1}{4\pi R^3}\epsilon_{ijk}x_i^L dx_j^L dx_k^L \tag{6.2}$$

and similarly ω^R. The normalization is chosen such that

$$\int_{S^2_{L,R}} \omega^{L,R} = 1 = \int_{S^2\times S^2} \omega^L \wedge \omega^R \tag{6.3}$$

so that ω^L, ω^R generate the integer cohomology $H^*(S^2 \times S^2, \mathbb{Z})$. Noting that ω is self-dual while $\tilde{\omega} := \omega^L - \omega^R$ is anti-selfdual, it follows immediately that both $F = 2\pi\omega$ and $F = 2\pi\tilde{\omega}$ are solutions of the abelian field equations. More generally, any

$$F^{(m_L,m_R)} = 2\pi m_L \omega^L + 2\pi m_R \omega^R \tag{6.4}$$

for any integers m_L, m_R is a solution. In bundle language, they correspond to products of 2 monopole bundles with connections and monopole number $m_{L,R}$ over $S^2_{L,R}$. Following the literature we will denote any such non-trivial solution as instanton.

6.1 Instantons and fluxons

We are interested in similar non-trivial solutions of the e.o.m. (3.4) in the fuzzy case. The monopole solutions on the fuzzy sphere S^2_N are given by representations λ_i^{N-m} of $su(2)$ of size $N-m$ [33], which lead to the classical monopole gauge fields in the commutative limit as shown in [7]. It is hence easy to guess that we will obtain solutions on $S^2_N \times S^2_N$ by taking products of these:

$$B_i^L = \alpha^L \, \lambda_i^{N-m_L} \otimes \mathbb{1}_{N-m_R}\,, \tag{6.5}$$

$$B_i^R = \alpha^R \, \mathbb{1}_{N-m_L} \otimes \lambda_i^{N-m_R} \tag{6.6}$$

where $\lambda_i^{N-m_{L,R}}$ are the $N-m_{L,R}$ dimensional generators of $su(2)$. It is not difficult to verify that these are solutions of (3.4) with $\alpha^{L,R} = 1 + \frac{m_{L,R}}{N}$ for $m_{L,R} \ll N$, with field strength

$$F_{iLjL} = -\frac{m^L}{2R^3}\epsilon_{ijk}x_k^L\,, \qquad F_{iRjR} = -\frac{m^R}{2R^3}\epsilon_{ijk}x_k^R\,, \qquad F_{iLjR} = 0\,, \tag{6.7}$$

while $B \cdot B - \frac{N^2-1}{4} \to 0$ as $N \to \infty$. This means that $F = -2\pi m^L\omega^L - 2\pi m^R \omega^R$ in the commutative limit, so that indeed

$$\int\limits_{S_2^{L,R}} \frac{F}{2\pi} = -m^{L,R}\,. \tag{6.8}$$

Notice that the Ansatz (6.6) implies that all matrices have size $\mathcal{N} = (N-m_L)(N-m_R)$, which is inconsistent if we require that $\mathcal{N} = N^2$ in order to have the original $S^2_N \times S^2_N$ vacuum. Therefore it appears that these solutions live in a different configuration space, similar as the commutative monopoles which live on different bundles. However, the situation is in fact more interesting: the above solutions can be embedded in the *same* configuration spaces of $N^2 \times N^2$ matrices as the vacuum solution if we combine them with other solutions, which have finite action in four dimensions.[5] They are in fact crucial to recover some of the known U(1) instantons in the limit $S^2_N \to \mathbb{R}^2_\theta$ resp. $S^2_N \times S^2_N \to \mathbb{R}^4_\theta$, as we will see. Consider the following Ansatz

$$B_i^{L,R} = \mathrm{diag}(d_{i,1}^{L,R}, \dots, d_{i,n}^{L,R})$$

in terms of diagonal matrices (ignoring the size of the matrices for the moment). These are solutions of (3.4) in two cases,

$$\sum_i d_{i,k}^{L,R} d_{i,k}^{L,R} = \begin{cases} \frac{N^2-3}{4}, & \text{type A} \\ 0, & \text{type B} \end{cases} \tag{6.9}$$

(i.e. $d_{i,k}^{L,R} = 0$ in type B). The associated field strength is

$$F_{iLjL} = \frac{\epsilon_{ijk}}{R^2}\,\mathrm{diag}(d_{k,1}^L, \dots, d_{k,n}^L)\,, \qquad F_{LR} = 0\,, \tag{6.10}$$

and a similar formula for F_{iRjR}. The constraint term is then $(B \cdot B - \frac{N^2-1}{4}) \to -\frac{1}{2}$ for type A, and $(B \cdot B - \frac{N^2-1}{4}) \to -\frac{N^2-1}{4}$ for type B in the large-N limit. In particular, only the type A solutions will have a finite contribution

$$S_{\text{fluxon}} = \frac{V}{g^2 \mathcal{N}} \left(\frac{n}{4R^4} + \frac{2n}{R^4} \frac{N^2-3}{4} \right) \quad \to \quad \frac{8\pi^2}{g^2}\, n \tag{6.11}$$

to the action,[6] which for $N \to \infty$ is only due to the field strength. We will see below that these type A solutions can be interpreted as a localized flux or vortex, and we will call them "fluxons" since they will reduce in a certain scaling limit to solutions on $\mathbb{R}^4_\theta$ which are sometimes denoted as such [34–36].

One can now combine these "fluxon" solutions with the monopole solutions (6.6) in the form

$$B_i^L = \begin{pmatrix} \alpha^L \lambda_i^{N-m_L} \otimes \mathbb{1}_{N-m_R} & 0 \\ 0 & \mathrm{diag}(d_{i,1}^L, \dots, d_{i,n}^L) \end{pmatrix},$$
$$B_i^R = \begin{pmatrix} \alpha^R\, \mathbb{1}_{N-m_L} \otimes \lambda_i^{N-m_R} & 0 \\ 0 & \mathrm{diag}(d_{i,1}^R, \dots, d_{i,n}^R) \end{pmatrix}. \tag{6.12}$$

These are now matrices of size $\mathcal{N} = (N - m_L)(N - m_R) + n$, which must agree with $\mathcal{N} = N^2$, say. This is clearly possible for

$$m_L = -m_R = m\,, \qquad n = m^2\,, \tag{6.13}$$

while for $m_L \neq -m_R$ the contribution from the fluxons would be infinite since $n = O(N)$. To understand these solutions, we can compute the gauge field from (3.5),

$$\mathcal{A}_i^L = \frac{1}{R}\left(B_i^L - \lambda_i^N \otimes \mathbb{1}_N\right) = \mathcal{A}_i^L(x^L, x^R)\,. \tag{6.14}$$

[5] as opposed to 2 dimensions, which is the reason why they were not considered in [7].

[6] A finite action can also be obtained for the type B solution using a slightly modified action (6.19), as discussed below.

To evaluate this, we first have to choose a gauge, i.e. a unitary transformation U for (6.12) which allows to express e.g. $\lambda_i^{N-m_L} \otimes \mathbb{1}_{N-m_R}$ in terms of $x_i^L \propto \lambda_i^N \otimes \mathbb{1}_N$ and $x_i^R \propto \mathbb{1}_N \otimes \lambda_i^N$. For example, in the case $m_L = -m_R = m$ this can be done using a unitary map

$$U: \; \mathbb{C}^{N-m} \otimes \mathbb{C}^{N+m} \oplus \mathbb{C}^{m^2} \to \mathbb{C}^N \otimes \mathbb{C}^N \,, \tag{6.15}$$

mapping a $(N-m) \times (N+m)$ matrix into a $N \times N$ matrix by trivially matching the upper-left corner in the obvious way, and fitting $\mathbb{C}^{m^2}$ into the remaining lower-right corner. With this being understood, one can write

$$\begin{aligned} R\mathcal{A}_i^L(x^L, x^R) &= (\alpha^L \lambda_i^{N-m} - \lambda_i^N) \otimes \mathbb{1}_{N+m} + \lambda_i^N \otimes (\mathbb{1}_{N+m} - \mathbb{1}_N) + (d - \text{terms}) \\ &= A_i^{(m)}(x^L) + \text{sing}(x_3^L = -R, x_3^R = -R) \end{aligned} \tag{6.16}$$

where $A_i^{(m)}(x^L)$ is indeed the gauge field of a monopole with charge m on S_L^2 in the large-N limit, as was checked explicitly in [7]. Here $\text{sing}(x_3^L = -R, x_3^R = -R)$ indicates a field localized at the "south pole" of S_L^2 and/or S_R^2 which becomes singular for large N. It originates both from "cutting and pasting" the bottom and right border of the above matrices using U (leading to singular gauge fields but regular field strength at the south poles), as well as the d-block (leading to a singular field strength). To see this recall that in general for the standard representation (A.2) of fuzzy spheres, entries in the lower-right block of the matrices correspond to functions localized at $x_3 = -R$, cp. (2.7). The gauge field near this singularity will be studied in more detail in section 7.3. The field strength is

$$F_{iLjL} = -\frac{m^L}{2R^3} \epsilon_{ijk} x_k^L + \epsilon_{ijk} \frac{1}{R^2} \sum_{i=1}^{n} d_{k,i}^L P_i \tag{6.17}$$

in the commutative limit, where P_i are projectors in the algebra of functions on $S_N^2 \times S_N^2$ of rank 1; recalling (2.7), they should be interpreted as delta-functions $P_i = \frac{V}{N^2} \delta^{(4)}(x_3 = -R)$. Similar formulae hold for $\mathcal{A}_i^L(x^L, x^R)$ and F_{iRjR}, while $F_{LR} = 0$.

We assumed above that these delta-functions are localized at the south poles $x_3^L = x_3^R = -R$. However, the location of these delta-functions can be chosen freely using gauge transformations. This can be seen by applying suitable successive gauge transformations using $N-k$-dimensional irreps of SU(2) for $k = 0, 1, \ldots, m-1$, which from the classical point of view all correspond to global rotations, successively moving the individual delta-peaks. Therefore the solution (6.12) should in general be interpreted as monopole on $S^2 \times S^2$ with monopole number $m_L = -m_R = m$, combined with a localized singular field strength characterized by its position and a vector $d_{k,i}^L$. We will see in section 7 that it becomes the "fluxon" solution in the planar limit $\mathbb{R}_\theta^4$; we therefore also call it a "fluxon".

The total action of these solutions (6.12) is the sum of the contributions from the monopole field plus the contribution from the fluxons (6.11), which both give the same contribution

$$S_{(m)} = \frac{4\pi^2}{g^2} \left(2m^2 + 2m^2\right) \tag{6.18}$$

in the large-N limit, using (6.13). The first term is due to the global monopole field (6.7), and the second term is the contribution of the fluxons through the localized field strength.

The interpretation of these solutions depends on the scaling limit $N \to \infty$ which we want to consider. We have seen that in the commutative limit keeping $R = \text{const}$, these solutions become commutative monopoles on $S^2 \times S^2$ with magnetic charges $m_L = -m_R$, plus additional localized "fluxon" degrees of freedom. For large R, the field strength of the monopoles vanishes, leaving only the localized fluxons. In particular, we will see in the following section that in the scaling limit $S^2_N \times S^2_N \to \mathbb{R}^4_\theta$ only the fluxons survive and become well-known solutions for gauge theory on $\mathbb{R}^4_\theta$. Away from this localized fluxon the gauge field becomes a flat connection, which is however topologically nontrivial. This is very interesting as it shows that one can indeed use these fuzzy spaces as regularization for gauge theory on $\mathbb{R}^{2n}_\theta$.

A final remark is in order: if we fix the size $\mathcal{N}$ of the matrices, only certain fluxon and monopole numbers are allowed, given by (6.13). Otherwise the number n of fluxons and hence the action would diverge with N. This can be seen as an interesting feature of our model: viewed as a regularization of gauge theory on $\mathbb{R}^4_\theta$, this points to possible subtleties of defining the admissible field configurations in infinite-dimensional Hilbert spaces and relations with topological terms in the action. On the other hand, we could accommodate the most general solutions including also type B solutions (6.9) by modifying the action similar as in [7]. For example,

$$S = \frac{1}{g^2}\int \left(\frac{4B_i^L B_i^L}{N^2R^4}\left(B_i^L B_i^L - \frac{N_L^2-1}{4}\right)^2 + \frac{4B_i^R B_i^R}{R^4}\left(B_i^R B_i^R - \frac{N_R^2-1}{4}\right)^2 + \frac{1}{2}F_{ia,jb}F_{ia,jb}\right) \tag{6.19}$$

leads to the same commutative action, but with a vanishing action for the Dirac string in the type B solutions.

6.2 Spherical branes

Consider the following solutions

$$\begin{aligned} B_i^L &= \begin{pmatrix} \alpha^L\, \lambda_i^{N-m} & 0 \\ 0 & \mathrm{diag}(d_{i,1},\dots,d_{i,m}) \end{pmatrix} \otimes \mathbb{1}_N\,, \\ B_i^R &= \mathbb{1}_N \otimes \lambda_i^N \end{aligned} \tag{6.20}$$

which are matrices of size $\mathcal{N} = N^2$. The corresponding field strength is

$$\begin{aligned} F_{iLjL} &= -\frac{m}{2R^3}\epsilon_{ijk}x_k^L + \epsilon_{ijk}\frac{1}{R^2}\sum_{i=1}^m d_{k,i}P_i \\ F_{RR} &= F_{LR} = 0 \end{aligned} \tag{6.21}$$

where P_i are projectors in the algebra of functions on S^2_L of rank 1 which should be interpreted as delta-functions $P_i = \frac{4\pi R^2}{N}\,\delta^{(2)}(x_3 = -R)$. In particular the gauge field A vanishes on S^2_R, while on S^2_L there is a monopole field together with a singularity at a point. This is similar to the fluxons on the previous section, but now only on S^2_L. This leads to the interpretation as 2-dimensional brane wrapping on S^2_R, located at a point on S^2_L. The action for these solutions is infinite. In the limit $S^2_N \times S^2_N \to \mathbb{R}^4_\theta$, the flux will be located at

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a 2-dimensional hyperplane. Such solutions for gauge theory on $\mathbb{R}^4_\theta$ were found in [37, 4], which would be recovered in the scaling limit $S^2_N \times S^2_N \to \mathbb{R}^4_\theta$ as discussed in section 7. In a similar way, we can interpret solutions with any m_L, m_R as branes wrapping on S^2_L and S^2_R.

7. Gauge theory on $\mathbb{R}^4_\theta$ from $S^2_{N_L} \times S^2_{N_R}$

We saw in section 2.1 that $\mathbb{R}^4_\theta$ can be obtained as a scaling limit of fuzzy $S^2_{N_L} \times S^2_{N_R}$. Here we will extend this scaling also to the covariant coordinates B_μ, thereby relating the gauge theory on $S^2_{N_L} \times S^2_{N_R}$ to that on $\mathbb{R}^4_\theta$ and hence providing a regularization for the latter. We will in particular relate the instanton solutions on these two spaces.

On noncommutative $\mathbb{R}^2_\theta$, all U(1)-instantons were constructed and classified in [4]. They can be interpreted as localized flux solutions, sometimes called fluxons. One can indeed recover these instantons from corresponding solutions on S^2_N, as we will show below. However since we are mainly interested in the 4-dimensional case here, we will only present the corresponding constructions on $S^2_{N_L} \times S^2_{N_R}$ resp. $\mathbb{R}^4_\theta$ here, without discussing the 2-dimensional case separately. It can be recovered in an obvious way from the considerations below.

The situation on $\mathbb{R}^4_\theta$ is more complicated, and there are different types of non-trivial U(1) "instanton" solutions on $\mathbb{R}^4_\theta$. Assuming that $\theta_{\mu\nu}$ is self-dual, there are two types of instantons: first, there exist straightforward generalizations of the localized "fluxon" solutions with self-dual field strength. These will be discussed in detail here, and we will show how these solutions can be recovered as scaling limits of the solutions (6.12) on $S^2_{N_L} \times S^2_{N_R}$. This is one of the main results of the present paper. In particular, the moduli of the fluxon solutions on $\mathbb{R}^4_\theta$ will be related to the free parameters $d_i^{L,R}$ in (6.12). This supports our suggestion to use gauge theory on $S^2_{N_L} \times S^2_{N_R}$ as a regularization for gauge theory on $\mathbb{R}^4_\theta$. However there are other types of U(1) instantons on $\mathbb{R}^4_\theta$ which were found through a noncommutative version of the ADHM equations [38–43], in particular anti-selfdual instantons which are much less localized than the fluxon solutions. To find the corresponding solutions on $S^2_{N_L} \times S^2_{N_R}$ is an interesting open challenge.

7.1 The action

The most general noncommutative $\mathbb{R}^4_\theta$ is generated by the coordinates subject to the commutation relations

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}\,, \tag{7.1}$$

where $\mu, \nu \in \{1, \ldots, 4\}$. Using suitable rotations, $\theta_{\mu\nu}$ can always be cast in the following form:

$$\theta_{\mu\nu} = \begin{pmatrix} 0 & \theta_{12} & 0 & 0 \\ -\theta_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_{34} \\ 0 & 0 & -\theta_{34} & 0 \end{pmatrix}.$$

We will assume that $\theta_{12} > 0$ and $\theta_{34} > 0$ for simplicity in this section. Then define

$$X_{1,2} := \sqrt{\frac{2\theta_{12}}{N_L}}\, B^L_{1,2}\,, \tag{7.2}$$

$$X_{3,4} := \sqrt{\frac{2\theta_{34}}{N_R}}\, B^R_{1,2}\,, \tag{7.3}$$

$$\phi^{L,R} := B_3^{L,R} - \frac{N_{L,R}}{2} + \frac{1}{N_{L,R}}((B_1^{L,R})^2 + (B_2^{L,R})^2)\,, \tag{7.4}$$

which should be interpreted as a blow-up near the north pole. In the scaling limit (2.17),

$$R^2 = \frac{1}{2}N_L\theta_{34} = \frac{1}{2}N_R\theta_{12} \quad \to \quad \infty \tag{7.5}$$

the X will become the covariant coordinates on the "tangential" $\mathbb{R}^4_\theta$ as $N_{L,R} \to \infty$, and ϕ remains an auxiliary field. To see this, we compute for the field strength

$$\begin{aligned}
\frac{1}{R^2}([B_1^L, B_1^R]) &= \frac{1}{\theta_{12}\theta_{34}}[X_1, X_3]\,, \qquad \text{etc.,} \\
\frac{1}{R^2}(B_1^L + i[B_2^L, B_3^L]) &= \sqrt{\frac{1}{\theta_{12}\theta_{34}R^2}}\,(X_1 + i[X_2, \phi^L] - \frac{i}{2\theta_{12}}[X_2, (X_1)^2]) \\
\frac{1}{R^2}(B_2^L + i[B_3^L, B_1^L]) &= \sqrt{\frac{1}{\theta_{12}\theta_{34}R^2}}\,(X_2 + i[X_1, \phi^L] - \frac{i}{2\theta_{12}}[X_1, (X_2)^2]) \\
\frac{1}{R^2}(B_3^L + i[B_1^L, B_2^L]) &= \frac{1}{\theta_{12}\theta_{34}}(\theta_{12} + i[X_1, X_2] + \frac{\theta_{12}\theta_{34}}{R^2}\phi_L - \frac{\theta_{12}\theta_{34}^2}{2R^4}((X_1)^2 + (X_2)^2))\,.
\end{aligned}$$

Analogous expressions hold for B_i^R. For the potential term we get

$$\begin{aligned}
\frac{1}{R^2}\left(B_i^L B_i^L - \frac{N_L^2 - 1}{4}\right) = \frac{1}{\theta_{34}}\phi^L + \frac{2}{R^2}\left((\phi^L)^2 + \frac{1}{4}\right) - \frac{1}{\theta_{12}R^2}\{\phi^L, (X_1)^2 + (X_2)^2\} + \\
+\frac{1}{\theta_{12}^2 R^2}((X_1)^2 + (X_2)^2)^2\,.
\end{aligned}$$

We immediately see that the only terms from action (3.1) involving $\phi^{L,R}$ are

$$\frac{1}{\theta_{34}^2}(\phi^L)^2 + \frac{1}{\theta_{12}^2}(\phi^R)^2 + O\left(\frac{1}{R}\right),$$

and therefore we can integrate them out in the limit $R \to \infty$. In the leading order in R the remaining terms give the standard action

$$S = -\frac{1}{2g^2\theta_{12}^2\theta_{34}^2}\int ([X_\mu, X_\nu] - i\theta_{\mu\nu})^2$$

for a gauge theory on $\mathbb{R}^4_\theta$ for general $\theta_{\mu\nu}$. The X_μ are interpreted as "covariant coordinates", which can be written as[7]

$$X_\mu := x_\mu + i\theta_{\mu\nu}A_\nu\,.$$

Hence the gauge fields A_μ describe the fluctuations around the vacuum. In particular, note that our regularization procedure clearly fixes the rank of the gauge group, unlike in the naive definition on $\mathbb{R}^d_\theta$ as discussed in [4]. The generalization to the U(n) case is obvious.

[7] We do not distinguish between upper and lower indices.

7.2 U(1) instantons on $\mathbb{R}^4_\theta$

The construction of instanton solutions for the two-dimensional noncommutative plane given in [4] can be easily generalized to the four-dimensional case. We shall recall and discuss these 4-dimensional "fluxon" solutions in some detail here, in order to understand the relation with the above solutions. To simplify the following formulas, we restrict our discussion from now on to the selfdual case

$$\theta_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\theta_{\rho\sigma}$$

and denote

$$\theta := \theta_{12} = \theta_{34};$$

the generalizations to the antiselfdual and the general case are obvious. Then the action for U(1) gauge theory on $\mathbb{R}^4_\theta$ reads

$$S = \frac{(2\pi)^2}{2g^2\theta^2}\,\mathrm{tr}(F_{\mu\nu}F_{\mu\nu}) \tag{7.6}$$

where

$$F_{\mu\nu} = i([X_\mu, X_\nu] - i\theta_{\mu\nu}) \tag{7.7}$$

is the field strength. In terms of the complex coordinates

$$x_{\pm L} := x_1 \pm ix_2\ ,\ x_{\pm R} := x_3 \pm ix_4\,,$$

the commutation relations (7.1) take the form

$$[x_{+a}, x_{-b}] = 2\theta\delta_{ab}\,, \qquad [x_{+a}, x_{+b}] = [x_{-a}, x_{-b}] = 0\,, \tag{7.8}$$

where $a, b \in \{L, R\}$. The Fock-space representation $\mathcal{H}$ of (7.8) has the standard basis

$$|n_1, n_2\rangle\,, \qquad n_1, n_2 \in \mathbb{N}\,,$$

with

$$\begin{aligned} x_{-L}|n_1,n_2\rangle &= \sqrt{2\theta}\sqrt{n_1+1}|n_1+1,n_2\rangle\,,\ \ x_{+L}|n_1,n_2\rangle = \sqrt{2\theta}\sqrt{n_1}|n_1-1,n_2\rangle \\ x_{-R}|n_1,n_2\rangle &= \sqrt{2\theta}\sqrt{n_2+1}|n_1,n_2+1\rangle\,,\ x_{+R}|n_1,n_2\rangle = \sqrt{2\theta}\sqrt{n_2}|n_1,n_2-1\rangle\,. \end{aligned}$$

Similarly, using the complex covariant coordinates $X_{\pm a}$

$$X_{\pm L} = X_1 \pm iX_2\ ,\ X_{\pm R} = X_3 \pm iX_4 \tag{7.9}$$

and the corresponding field strength

$$F_{\alpha a,\beta b} = [X_{\alpha a}, X_{\beta b}] - 2\theta\varepsilon_{\alpha\beta}\delta_{ab}$$

with $a, b \in \{L, R\}$ and $\alpha, \beta \in \{+, -\}$, the action (7.6) can be written in the form

$$S = \frac{\pi^2}{g^2\theta^2}\,\mathrm{tr}\Big(\sum_a F_{+a,-a}F_{+a,-a} - \sum_{a,b} F_{+a,+b}F_{-a,-b}\Big)\,.$$

Then the equations of motion are given by:

$$\sum_{a,\alpha}[X_{\alpha a}, (F_{\alpha a,\beta b})^\dagger] = 0\,. \tag{7.10}$$

Let us consider a finite dimensional subvectorspace V_n of $\mathcal{H}$ of dimension n, which we can assume (using a unitary gauge transformation) to be spanned by a finite set of vectors $|n_1, n_2\rangle \in \mathcal{H}$,

$$V_n = \langle \{|i_k, j_k\rangle\,;\ k = 1, \ldots, n\}\rangle\,. \tag{7.11}$$

Following [4] one finds solutions to the equations of motion given by[8]

$$X_{+L}^{(n)} := Sx_{+L}S^\dagger + \sum_{k=1}^{n} \gamma_k^L |i_k, j_k\rangle\langle i_k, j_k| \tag{7.12}$$

$$X_{+R}^{(n)} := Sx_{+R}S^\dagger + \sum_{k=1}^{n} \gamma_k^R |i_k, j_k\rangle\langle i_k, j_k|\,. \tag{7.13}$$

Here $\gamma_k^{L,R} \in \mathbb{C}$ determine the position of the fluxons, and S denotes a partial isometry from $\mathcal{H}$ to $\mathcal{H}\backslash V_n$ with $S^\dagger S = \mathbb{1}$, $SS^\dagger = \mathbb{1} - P_{V_n}$, where

$$P_{V_n} := \sum_{k=1}^{n} |i_k, j_k\rangle\langle i_k, j_k|$$

is the projection operator onto the subspace V_n. The field strength $F_{\mu\nu}$ for this solution is

$$F_{\mu\nu} = P_{V_n}\theta_{\mu\nu}\,.$$

In particular, the action corresponding to the instanton solution (7.12), (7.13) is proportional to the dimension of the subspace V_n

$$S[X_{\pm a}^{(n)}] = \frac{8\pi^2}{g^2}\,\mathrm{tr}(P_{V_n}) = \frac{8\pi^2}{g^2}\,n\,.$$

We will see in the next section that this class of solutions can be reproduced by instanton solutions (6.12) on $S^2_{N_L} \times S^2_{N_R}$ in a suitable scaling limit. Let us stress again that this is only one class of U(1)-instanton solutions for $\mathbb{R}^4_\theta$ which is called “fluxons”, since they can be interpreted as localized flux. The localization can be seen as follows: recall [44] that the above projection operators can be represented on the space of commutative functions (using a normal-ordering prescription) as

$$|k^1, k^2\rangle\langle k^1, k^2| \cong \frac{1}{k^1! k^2!}\left(\frac{x^{-L}}{\sqrt{2\theta}}\right)^{k^1}\left(\frac{x^{+L}}{\sqrt{2\theta}}\right)^{k^1}\left(\frac{x^{-R}}{\sqrt{2\theta}}\right)^{k^2}\left(\frac{x^{+R}}{\sqrt{2\theta}}\right)^{k^2} e^{-\frac{x^{+L}x^{-L}}{2\theta} - \frac{x^{+R}x^{-R}}{2\theta}}\,.$$

Hence the above field strengths $F_{\mu\nu} = P_{V_n}\theta_{\mu\nu}$ are superpositions of Gauss-functions which are localized in a region in space of size $\sqrt{\theta}$.

[8] Note that $[X_{+L}^{(n)}, X_{+R}^{(n)}] = [X_{+L}^{(n)}, X_{-R}^{(n)}] = [X_{-L}^{(n)}, X_{+R}^{(n)}] = [X_{-L}^{(n)}, X_{-R}^{(n)}] = 0$.

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7.3 Instantons on $\mathbb{R}^4_\theta$ from $S^2_N \times S^2_N$

With the scaling limit of section 7.1, the gauge theory on $S^2_N \times S^2_N$ provides us with a regularization for the gauge theory on $\mathbb{R}^4_\theta$. Of course, such a regularization might affect the topological features of the theory, an effect we want to investigate in this section. For this, we will map the topologically nontrivial solutions found in section 6 on $S^2_N \times S^2_N$ to $\mathbb{R}^4_\theta$.

Consider again the solutions (6.12) that combine the fluxon solutions with the monopoles, with the fluxons at the north pole instead of the south pole because we want to study their structure. Their scaling limit as in (7.2) gives

$$X_i = \sqrt{\frac{2\theta}{N}} \begin{pmatrix} \mathrm{diag}(d^L_{i,1}, \dots, d^L_{i,n}) & 0 \\ 0 & \alpha^L \lambda_i^{N-m} \otimes \mathbb{1} \end{pmatrix}, \tag{7.14}$$

$$X_{i+2} = \sqrt{\frac{2\theta}{N}} \begin{pmatrix} \mathrm{diag}(d^R_{i,1}, \dots, d^R_{i,n}) & 0 \\ 0 & \alpha^R \mathbb{1} \otimes \lambda_i^{N+m} \end{pmatrix} \tag{7.15}$$

for $i = 1,2$. Recalling that the rescaled $\lambda_{1,2}$ on $S^2_{N_L} \times S^2_{N_R}$ become the $x_\pm$'s on $\mathbb{R}^4_\theta$ in the scaling limit

$$\sqrt{\frac{2\theta}{N}} (\lambda_1^{L,R} \pm i \lambda_2^{L,R}) \to x_{\pm L,R}\,,$$

we see that (7.14) and (7.15) become the instantons (7.12), (7.13) on $\mathbb{R}^4_\theta$,

$$X_1 + iX_2 \to X^{(n)}_{+L} = S x_{+L} S^\dagger + \sum_{k=1}^{n} \gamma^L_k |i_k, j_k\rangle\langle i_k, j_k|\,, \tag{7.16}$$

$$X_3 + iX_4 \to X^{(n)}_{+R} = S x_{+R} S^\dagger + \sum_{k=1}^{n} \gamma^R_k |i_k, j_k\rangle\langle i_k, j_k|\,. \tag{7.17}$$

Here the (d_i)-block acting on a basis $|i_k, j_k\rangle$ of $V_n \subset \mathcal{H} \cong \mathbb{C}^{\mathcal{N}}$ becomes the projector part of (7.16), (7.17) with

$$\begin{aligned} \sqrt{\frac{2\theta}{N}}\, d^{L,R}_{1,k} &\to \mathrm{Re}\gamma^{L,R}_k\,, \\ \sqrt{\frac{2\theta}{N}}\, d^{L,R}_{2,k} &\to \mathrm{Im}\gamma^{L,R}_k\,, \end{aligned} \tag{7.18}$$

and the monopole block becomes $S x_+ S^\dagger$ where S is a partial isometry from $\mathcal{H}$ to $\mathcal{H} \backslash V_n$. Note that we can recover any value for the γ's in this scaling, solving the constraint $d_i d_i = \frac{N^2-3}{4}$ for $d_3 \sim \frac{N}{2}$. Therefore the full moduli space of the fluxon solutions (7.12), (7.13) on $\mathbb{R}^4_\theta$ can be recovered in this way. Furthermore, the meaning of the parameters $\gamma^{L,R}$ is easy to understand in our approach: Note first that using a rotation (which acts also on the indices) followed by a gauge transformation, the d_i can be fixed to be radial at the north pole, $d_i^{L,R} \sim (0,0,N/2)$. This is a fluxon localized at the north pole. Now apply a "translation" at the north pole, which corresponds to a suitable rotation on the sphere. Rotating the vector $d_i^{L,R}$ in the scaling limit amounts to a translation of the $\gamma_k^{L,R}$ according to (7.18), which therefore parametrize the position of the fluxons.

It has been noted [2] that the $Sx_+S^\dagger$ correspond to a pure (but topologically nontrivial) gauge, which can qualitatively be seen already in two dimensions. There, the isomorphism $S : |k\rangle \to |k+n\rangle$ is basically $(\frac{x_-}{\sqrt{x_-x_+}})^n \sim (\frac{x-iy}{r})^n \sim e^{in\varphi}$ and therefore the gauge field $A_i = S\partial_i S^\dagger$ has a winding number n. The topological nature of the $Sx_+S^\dagger$ is even more evident in our setting, as they are the limit of the monopole solutions (6.5), (6.6) on $S^2_N \times S^2_N$. Moreover, note that their contribution to the action (6.18) survives the scaling: even though the field strength vanishes as $R \to \infty$, the integral gives a finite contribution equal to the contribution of the fluxon part. This topological "surface term" is usually omitted in the literature on $\mathbb{R}^4_\theta$, but becomes apparent in the regularized theory.

So it seems that we recovered all the instantons of section 7.2, but in fact there is an important detail that we haven't discussed jet. It is the embedding of the n-dimensional fluxons and the $(N-m)(N+m)$-dimensional monopole solutions into the N^2-dimensional matrices of the ground state. Such an embedding is clearly only possible for $n = m^2$. This means that the regularized theory has some kind of "superselection rule" for the dimension of the allowed instantons, a rule that did not exist in the unregularized theory.[9]

One way to allow arbitrary instanton numbers is to allow the size $\mathcal{N}$ of the matrices to vary. However, this is less satisfactory as it destroys the unification of topological sectors which is a beautiful feature of noncommutative gauge theory. On the other hand, the type B solutions (6.9) together with the changed action (6.19) might allow the construction of the missing instantons. The idea is to fill up the unnecessary $m^2 - n$ places with $d_i = 0$. The changed action would not suppress such solutions any more, and in fact they would not even contribute to the action. This amounts to adding a discrete sector to the theory which accommodates these type B solutions, but decouples from the rest of the model. Whether or not one wants to do this appears to be a matter of choice. This emphasizes again the importance of a careful regularization of the theory. It would be very interesting to see what happens in other regularizations e.g. using gauge theory on noncommutative tori or fuzzy $\mathbb{C}P^2$.

8. Fermions

8.1 The commutative Dirac operator on $S^2 \times S^2$

To find a form of the commutative Dirac operator on $S^2 \times S^2$ which is suitable for the fuzzy case, one can generalize the approach of [45] for S^2, which is carried out in detail in appendix E.3: One can write the flat SO(6) Dirac operator D_6 in 2 different forms, using spherical coordinates of the spheres and also using the usual flat euclidean coordinates. Then one can relate D_6 with the curved four-dimensional Dirac operator D_4 on $S^2 \times S^2$ in the same spherical coordinates. This leads to an explicit expression for D_4 involving only the angular momentum generators, which is easy to generalize to the fuzzy case. The

[9]Note that this is different in two dimensions. There, a rank n fluxon can be combined with a $(N-n)$-dimensional monopole block and all the instantons on $\mathbb{R}^2_\theta$ can be recovered. Furthermore, the actions for the fluxons and the monopoles scale differently with N. Therefore the action for the monopoles vanishes in the scaling limit that produces a gauge theory on $\mathbb{R}^2_\theta$ with rescaled coupling constant.

result is rather obvious and easy to guess:

$$D_4 = \Gamma^\mu J_\mu + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \Gamma^\mu J_\mu + 2C_0\,, \tag{8.1}$$

which is clearly a SO(3) $\times$ SO(3)-covariant hermitean first-oder differential operator. Here Γ^μ generate the SO(6) Clifford algebra (4.10), C_0 is defined in (4.13), and we put $R = 1$ for simplicity here. However this Dirac operator is reducible, acting on 8-dimensional spinors Ψ_8 corresponding to the SO(6) Clifford algebra. Hence Ψ_8 should be a combination of two independent 4-component Dirac spinors on the 4-dimensional space $S^2 \times S^2$. To see this, we will construct explicit projectors projecting onto these 4-dimensional spinors, and identify the appropriate 4-dimensional chirality operators. This will provide us with the desired physical Dirac or Weyl fermions.

8.1.1 Chirality and projections for the spinors

There are 3 obvious operators which anti-commute with D_4. One is the usual 6-dimensional chirality operator

$$\Gamma := i\Gamma^L_1\Gamma^L_2\Gamma^L_3\Gamma^R_1\Gamma^R_2\Gamma^R_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{8.2}$$

which satisfies

$$\{D_4, \Gamma\} = 0\,, \qquad \Gamma^\dagger = \Gamma\,, \qquad \Gamma^2 = 1\,. \tag{8.3}$$

The 8-component spinors Ψ_8 split accordingly into two 4-component spinors $\Psi_8 = (\frac{\psi_\alpha}{\psi_{\overline{\beta}}})$, which transform as (4) resp. $(\overline{4})$ under $so(6) \cong su(4)$; recall the related discussion in section 4. The other operators of interest are

$$\chi_L = \Gamma^{iL} x_{iL} \qquad \text{and} \chi_R = \Gamma^{iR} x_{iR}\,.$$

They preserve SO(3) $\times$ SO(3) $\subset$ SO(6), and satisfy

$$\{D_4, \chi_{L,R}\} = 0 = \{\chi_L, \chi_R\}$$

as well as

$$\chi^2_{L,R} = 1\,.$$

We will also use

$$\chi = \frac{1}{\sqrt{2}}\,\Gamma^\mu x_\mu = \frac{1}{\sqrt{2}}\,(\chi_L + \chi_R) \tag{8.4}$$

which satisfies similar relations. This means that

$$P_\pm = \frac{1}{2}(1 \pm i\chi_L\chi_R) \tag{8.5}$$

with

$$P_\pm^2 = P_\pm\,, \qquad P_+ + P_- = 1 \qquad \text{and} \qquad P_+P_- = 0 \tag{8.6}$$

are hermitean projectors commuting with the Dirac operator on $S^2 \times S^2$ as well as with Γ,

$$P_\pm^\dagger = P_\pm \qquad \text{and} \qquad [P_\pm, D_4] = [P_\pm, \Gamma] = 0\,. \tag{8.7}$$

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Therefore they project onto subspaces which are preserved by D_4 and Γ, and are invariant under $SO(3) \times SO(3)$. Hence the spinor lagrangian can be written as

$$\Psi_8^\dagger D_4 \Psi_8 = \Psi_+^\dagger D_4 \Psi_+ + \Psi_-^\dagger D_4 \Psi_-$$

involving two Dirac spinors $\Psi_\pm = P_\pm \Psi_8$. In order to get one 4-component Dirac spinor, we can e.g. impose the constraint

$$P_+ \Psi_8 = \Psi_8 \,, \tag{8.8}$$

or equivalently give one of the two components a large mass, by adding a term

$$M_- \Psi_8^\dagger P_- \Psi_8 \tag{8.9}$$

to the action with $M_- \to \infty$. The physical chirality operator is now identified using (8.7) and (8.3) as Γ acting on Ψ_+. It can be used to define 2-component Weyl spinors on $S^2 \times S^2$.

To make the above more explicit, consider the north-pole of the spheres, i.e.

$$x_L = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad x_R = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} .$$

In the basis (4.9) for the Clifford algebra we then get explicitly

$$P_\pm = \frac{1}{2}\left(1 \pm i \begin{pmatrix} -\gamma_L^1 \gamma_R^1 & 0 \\ 0 & \gamma_L^1 \gamma_R^1 \end{pmatrix} \right) = \frac{1}{2}(1 \pm \sigma_3 \otimes \sigma_3 \otimes \sigma_3) \,.$$

This means that

$$P_+ = \mathrm{diag}(1,0,0,1,0,1,1,0)$$

projects onto a 4-dimensional subspace exactly as expected.

8.2 Gauged fuzzy Dirac and chirality operators

To find fuzzy analogues of (8.1) and (8.4) coupled to the gauge fields, we recall the connection between the gauge theory on $S^2 \times S^2$ and the $SO(6)$ Gamma matrices established in section 4. In the spirit of that section a natural fuzzy spinor action would involve

$$\Psi^\dagger C \Psi \,, \tag{8.10}$$

where Ψ is now a $8\mathcal{N} \times \mathcal{N}$-matrix (with Grassman entries). Of course, (8.10) does not have the appropriate commutative limit, but we can split C into a fuzzy Dirac operator $\widehat{D}$ and the operator $\widehat{\chi}$ defined by

$$\widehat{\chi}\Psi = \frac{\sqrt{2}}{N}\left(\Gamma^\mu \Psi \lambda_\mu - C_0 \Psi\right), \tag{8.11}$$

which generalizes (8.4); we used here the definition (4.12),(4.13) of C_0. This operator satisfies

$$\widehat{\chi}^2 = 1 \,,$$

and reduces to (8.4) in the commutative limit. Note also that $\widehat{\chi}$ commutes with gauge transformations, since the coordinates λ_μ are acting from the right in (8.11). Setting

$$\widehat{J}_\mu \Psi = [\lambda_\mu, \Psi]\,,$$

we get for the fuzzy Dirac operator

$$\widehat{D} = C - \frac{N}{\sqrt{2}}\,\widehat{\chi} = \Gamma^\mu(\widehat{J}_\mu + A_\mu) + 2C_0 = \Gamma^\mu \widehat{\mathcal{D}}_\mu + 2C_0\,. \tag{8.12}$$

Here[10]

$$\widehat{\mathcal{D}}_\mu := \widehat{J}_\mu + A_\mu \tag{8.13}$$

is a covariant derivative operator, i.e. $U\widehat{\mathcal{D}}_\mu \psi = \widehat{\mathcal{D}}'_\mu U\psi$ which is easily verified using (3.6). This $\widehat{D}$ clearly has the correct classical limit (8.1) for vanishing A, and the gauge fields are coupled correctly. In particular, this definition of $\widehat{D}$ applies also to the topologically non-trivial solutions of section 6 without any modifications. Moreover, the chirality operator Γ as defined in (8.2) anti-commutes with $\widehat{D}$ also in the fuzzy case,

$$\{\widehat{D}, \Gamma\} = 0\,. \tag{8.14}$$

In particular there is no need to consider e.g. fuzzy Ginsparg-Wilson operators as in the 2-dimensional case [46 – 48]. However, the anticommutator of $\widehat{D}$ and $\widehat{\chi}$ no longer vanishes. We find

$$\{\widehat{D}, \widehat{\chi}\} = -\frac{\sqrt{2}}{N}\Big(2(\lambda_\mu + A_\mu)\widehat{J}_\mu - 2A_\mu\lambda_\mu + \{\Gamma^\mu, C_0\}\widehat{\mathcal{D}}_\mu + 2\Big) = O\Big(\frac{1}{N}\Big)\,, \tag{8.15}$$

since $x_\mu J_\mu = O(1/N)$ and $x_\mu A_\mu = O(1/N)$ using (3.9). Furthermore, using some identities given at the beginning of section 4 we obtain for $\widehat{D}^2\psi$:

$$\begin{aligned}\widehat{D}^2\psi \;&=\; (\Sigma^{\mu\nu}F_{\mu\nu} + \widehat{\mathcal{D}}_\mu\widehat{\mathcal{D}}_\mu + \{\Gamma^\mu, C_0\}\widehat{\mathcal{D}}_\mu + 2)\psi \\ &=: (\Sigma^{\mu\nu}F_{\mu\nu} + \widehat{\Box} + 2)\psi\,, \end{aligned} \tag{8.16}$$

defining the covariant 4-dimensional laplacian $\widehat{\Box}$ acting on the spinors. This corresponds to the usual expression for $\widehat{D}^2$ on curved spaces, and the constant 2 is due to the curvature scalar. Since $\widehat{D}^2$ and $\Sigma^{\mu\nu}F_{\mu\nu}$ are both hermitean and commute with Γ and $\widehat{P}_\pm$ as defined in (8.18) in the large-N limit, it follows that $\widehat{\Box}$ satisfies these properties as well. Note that (8.16) can also be written as

$$(\widehat{D} - C_0)^2 = \Sigma^{\mu\nu}F_{\mu\nu} + \widehat{\mathcal{D}}_\mu\widehat{\mathcal{D}}_\mu + \frac{1}{2}\,, \tag{8.17}$$

which might suggest to interpret $\widehat{\mathcal{D}}_\mu\widehat{\mathcal{D}}_\mu$ as covariant laplacian; however this is not correct since $\widehat{\mathcal{D}}_\mu\widehat{\mathcal{D}}_\mu$ does not commute with the projections $\widehat{P}_\pm$ (8.18) even in the commutative limit. The reason for this is our formulation using spinors based on the SO(6) Clifford algebra rather than SO(4) spinors and comoving frames. The corresponding projections to physical Dirac- or Weyl-spinors in the fuzzy case will be discussed next.

[10] We set $R = 1$ in this section for simplicity.

8.2.1 Projections for the fuzzy spinors

For the fuzzy case, we can again consider the following projection operators

$$\begin{aligned}\widehat{\chi}_L\Psi &= \frac{2}{N}(\Gamma^{iL}\Psi\lambda_{iL} + C_0^L\Psi)\,,\\ \widehat{\chi}_R\Psi &= \frac{2}{N}(\Gamma^{iR}\Psi\lambda_{iR} + C_0^R\Psi)\end{aligned}$$

which satisfy

$$\widehat{\chi}^2_{L,R} = 1\,, \qquad \{\widehat{\chi}_L, \widehat{\chi}_R\} = 0\,.$$

This implies $(\widehat{\chi}_L\widehat{\chi}_R)^2 = -1$, and we can write down the following projection operators

$$\widehat{P}_\pm = \frac{1}{2}(1 \pm i\widehat{\chi}_L\widehat{\chi}_R) \tag{8.18}$$

which have the classical limit (8.5) and the properties (8.6). However, the projector no longer commutes with the fuzzy Dirac operator (8.12):

$$\begin{aligned}[\widehat{D}, \widehat{\chi}_L\widehat{\chi}_R] &= \{\widehat{D}, \widehat{\chi}_L\}\widehat{\chi}_R - \widehat{\chi}_L\{\widehat{D}, \widehat{\chi}_R\}\\ &= -\frac{2}{N}\Big((2(\lambda_{iL} + \mathcal{A}_{iL})\widehat{J}_{iL} - 2A_{iL}\lambda_{iL} + 2C_0^L\,\Gamma^{iL}\widehat{\mathcal{D}}_{iL} + 1)\widehat{\chi}_R -\\ &\qquad - \widehat{\chi}_L(2(\lambda_{iR} + \mathcal{A}_{iR})\widehat{J}_{iR} - 2A_{iR}\lambda_{iR} + 2C_0^R\,\Gamma^{iR}\widehat{\mathcal{D}}_{iR} + 1)\Big)\,,\end{aligned}$$

which only vanishes for $N \to \infty$ and tangential A_μ (3.9). To reduce the degrees of freedom to one Dirac 4-spinor, we should therefore add a mass term

$$M_-\Psi_8^\dagger\widehat{P}_-\Psi_8 \tag{8.19}$$

which for $M_- \to \infty$ suppresses one of the spinors, rather than impose an exact constraint as in (8.8). This is gauge invariant since $\widehat{P}_\pm$ commutes with gauge transformations,

$$\widehat{P}_\pm\psi \to U\widehat{P}_\pm\psi\,.$$

The complete action for a Dirac fermion on fuzzy $S^2_N \times S^2_N$ is therefore given by

$$S_{\text{Dirac}} = \int \Psi_8^\dagger(\widehat{D} + m)\Psi_8 + M_-\Psi_8^\dagger\widehat{P}_-\Psi_8 \tag{8.20}$$

with $M_- \to \infty$. The physical chirality operator is given by Γ (8.2), which allows to consider Weyl spinors as well.

9. Conclusion and outlook

We have constructed U(n) gauge theory on fuzzy $S^2_N \times S^2_N$ as a multi-matrix model. The model is completely finite, and can be considered as a regularization either of Yang-Mills on the commutative $S^2 \times S^2$, or on the noncommutative $\mathbb{R}^4_\theta$ in a suitable scaling limit. The quantization is defined by a finite "path" integral over the matrix degrees of freedom, which is convergent due to the constraint term. A gauge-fixed action with BRST symmetry is also provided. We then discussed some topologically non-trivial solutions in the U(1) case, which reduce to the known "fluxon" solutions on $\mathbb{R}^4_\theta$ in the appropriate scaling limit, reproducing the full moduli space. On $S^2_N \times S^2_N$ they arise as localized flux tubes together with a monopole background field. This provides a very clean non-perturbative definition of noncommutative gauge theory with fixed rank of the gauge group U(n), and a simple description of instantons as solutions of the equation of motion in one single configuration space. Furthermore, we have shown how charged fermions in the fundamental representation can be coupled to the gauge field, by defining a suitable Dirac operator $\widehat{D}$. This is easily extended to Weyl fermions using a chirality operator which exactly anticommutes with $\widehat{D}$. All this supports the programme to formulate and study physically interesting models on noncommutative spaces.

There are many interesting conclusions and applications to be explored. One crucial feature is the fact that the model is completely regularized, i.e. the quantization is well-defined without any divergences for finite N. This should allow to study suitable scaling limits in N in a well-defined framework, and the emergence of an interesting low-energy limit which could be either commutative or noncommutative. Such a matrix regularization is very interesting in view of the UV/IR mixing, which indicates a close relationship between NC field theory and matrix models. For example, one might try to extend the results in [49] in this context. We also explored some alternative formulations using "collective matrices" based on SO(6). Such formulations are possible only in the noncommutative case, and lead to the hope that new non-perturbative techniques in the spirit of random matrix theory may be developed along these lines.

Another important aspect is the coupling to fermions, which could be extended to scalars and allows to study spontaneous symmetry breaking and the possible generation of other gauge groups in the low-energy limit. Finally, a detailed comparison with other finite models of NC gauge theory in 4 dimensions such as [22, 24, 25] would be very desirable, to see which features are generic and which are model-dependent.

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A. The standard representation of the fuzzy sphere

The irreducible N-dimensional representation of the $su(2)$ algebra λ_i (2.3) is given by

$$(\lambda_3)_{kl} = \delta_{kl}\,\frac{N+1-2k}{2}\,, \tag{A.1}$$

$$(\lambda_+)_{kl} = \delta_{k+1,l}\sqrt{(N-k)k}, \tag{A.2}$$

where $k, l = 1, \ldots, N$ and $\lambda_\pm = \lambda_1 \pm i\lambda_2$.

B. Alternative formulation using $4\mathcal{N} \times 4\mathcal{N}$ matrices

Let us rewrite the action (4.17) in terms of the $4\mathcal{N} \times 4\mathcal{N}$ matrices B_L, B_R (4.8). Noting that

$$C_L C_R + C_R C_L = \begin{pmatrix} -[B_L, B_R] & 0 \\ 0 & [B_L, B_R] \end{pmatrix} \tag{B.1}$$

we can rewrite S_6 (4.15) as

$$S_6 = 2\mathrm{Tr}\left(B_L^2 - B_R^2 - \frac{N^2}{2}\right)^2 + 2\mathrm{Tr}\left([B_L, B_R]^2\right), \tag{B.2}$$

where the trace is now over $4\mathcal{N} \times 4\mathcal{N}$ matrices. Similarly

$$S_{\mathrm{break}} = -2\mathrm{Tr}\left(B_L^2 - \frac{N^2}{4}\right)\left(-B_R^2 - \frac{N^2}{4}\right) \tag{B.3}$$

and combined we recover (3.1) as

$$S = S_6 - 2S_{\mathrm{break}} = 2\mathrm{Tr}\left((B_L^2 - \frac{N^2}{4})^2 + (-B_R^2 - \frac{N^2}{4})^2 + [B_L, B_R]^2\right). \tag{B.4}$$

This looks like a 2-matrix model, however the degrees of freedom B_L, B_R are still very much constrained and span only a small subspace of the $4\mathcal{N} \times 4\mathcal{N}$ matrices. We would like to find an intrinsic characterization without using the γ_μ explicitly. One possibility is to choose the γ_μ to be completely anti-symmetric matrices, see appendix D. However this does not extend to B, since the B_μ should be hermitean and not necessarily symmetric, and moreover the γ_μ are not hermitean (the conjugate being the intertwiner $(6) \subset (\overline{4}) \otimes (\overline{4})$). Another possibility is provided by the following representation of the γ-matrices:

$$\gamma_L^i = \sigma^i \otimes \mathbb{1}_{2\times 2}, \qquad \gamma_R^i = \mathbb{1}_{2\times 2} \otimes i\sigma^i. \tag{B.5}$$

They satisfy the relations (4.4)–eqg-L-R, but are not antisymmetric. Now note that

$$\gamma_R^i = iP\gamma_L^i P \tag{B.6}$$

where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}(1 + \sigma^i \otimes \sigma^i) \tag{B.7}$$

permutes the two tensor factors and satisfies

$$P^2 = 1. \tag{B.8}$$

Therefore we can characterize the degrees of freedom in terms of 2 hermitean $2\mathcal{N} \times 2\mathcal{N}$ matrices

$$X_L = B_L^i \sigma_i + \frac{1}{2}, \qquad X_R = B_R^i \sigma_i + \frac{1}{2} \tag{B.9}$$

which are arbitrary up to the constraint that $X^0_{L,R} = \frac{1}{2}$. Then

$$B_L = X_L \otimes \mathbb{1}_{2\times 2}\,, \qquad B_R = iP(X_R \otimes \mathbb{1}_{2\times 2})P\,; \tag{B.10}$$

they could be extracted from a single complex matrix $\tilde{B} = (X_L + iX_R) \otimes \mathbb{1}_{2\times 2}$. Furthermore, matrices of the form $X \otimes \mathbb{1}_{2\times 2}$ are characterized through their spectrum, which is doubly degenerate; indeed any such hermitean matrix can be cast into the above form using suitable unitary SU($4\mathcal{N}$) transformations. Similarly, P can also be characterized intrinsically: any matrix P written as

$$P = P_0 \otimes \mathbb{1}_{2\times 2} + P_i \otimes \sigma^i \tag{B.11}$$

which satisfies the constraints

$$P_0 = \frac{1}{2}\,, \qquad P^2 = \mathbb{1} \tag{B.12}$$

is given by (B.7) up to an irrelevant unitary transformation $U \otimes \mathbb{1}$. We could therefore write down the action (B.4) in terms of three matrices $B_L, -iPB_RP$ and P, all of which are characterized by their spectrum and constraints of the form $(..)_0 = \frac{1}{2}$. The hope is that such a reformulation may allow to apply some of the powerful methods from random matrix theory, in the spirit of [7]. However we will leave this for future investigations.

C. Stability analysis of the SO(6)-invariant action (4.15)

Consider the action (4.15). We will split off the radial degrees of freedom for large-N by setting $R = 1$ and[11]

$$B_{iL} = \lambda_{iL} + A_{iL} = \lambda_{iL} + \mathcal{A}_{iL} + x_{iL}\Phi_L$$

requiring that $\lambda_{iL}\mathcal{A}_{iL} = 0$, and similarly for B_{iR}, The stability of our geometry will depend on the behavior of Φ^L and Φ^R. We calculate that

$$B_\mu B_\mu - \frac{N^2-1}{2} = N(\Phi_L + \Phi_R) + \Phi_L\Phi_L + \Phi_R\Phi_R + \mathcal{A}_\mu\mathcal{A}_\mu - [\lambda_\mu, \mathcal{A}_\mu] + O\left(\frac{1}{N}\right),$$

where we used that $\lambda_{ia}\mathcal{A}_{ia} = 0$ and therefore both $\mathcal{A}_{ia}x_{ia} = O(1/N)$ and $\mathcal{A}_{ia}[\lambda_{ia}, \cdot\,] = O(1/N)$ for $a = L, R$. Setting

$$\begin{aligned}\Phi_L + \Phi_R &= \Phi_1\,,\\ \Phi_L - \Phi_R &= \Phi_2\end{aligned}$$

we get

$$B_\mu B_\mu - \frac{N^2-1}{2} = N\Phi_1 + \Phi_1\Phi_1 + \Phi_2\Phi_2 + \mathcal{A}_\mu\mathcal{A}_\mu - [\lambda_\mu, \mathcal{A}_\mu] + O\left(\frac{1}{N}\right). \tag{C.1}$$

In the limit $N \to \infty$ we can integrate out Φ_1, as it acquires an infinite mass. Alternatively we can rescale Φ_1 by setting $\phi_1 = \frac{1}{N}\Phi_1$. Then, all the terms involving ϕ_1 but the first one in (C.1) will be of order $\frac{1}{N}$ and we can equally integrate out ϕ_1.

[11] The fact that this leads to non-hermitian fields for finite N is not essential here.

The terms from

$$F_\mu F_\mu - [B_{iL}, B_{iR}]^2$$

involving the remaining Φ_2 will be (in the limit $N \to \infty$)

$$\frac{1}{2}\Phi_2\Phi_2 - J_\mu(\Phi_2)J_\mu(\Phi_2) - F_{iL}x_{iL}\Phi_2 + F_{iR}x_{iR}\Phi_2$$

with the tangential derivatives $J_{ia} = -i\epsilon_{ijk}x_{ja}\partial_{ka}$. Calculating that

$$J_\mu\Phi_2 J_\mu\Phi_2 = -\partial_\mu\Phi_2\partial_\mu\Phi_2 - x_{iL}\partial_{iL}\Phi_2 x_{jL}\partial_{jL}\Phi_2 - x_{iR}\partial_{iR}\Phi_2 x_{jR}\partial_{jR}\Phi_2$$

and using partial integration under the integral this gives

$$\frac{1}{2}\Phi_2\Phi_2 - \Phi_2\partial_\mu\partial_\mu\Phi_2 - x_{iL}\partial_{iL}\Phi_2 x_{jL}\partial_{jL}\Phi_2 - x_{iR}\partial_{iR}\Phi_2 x_{jR}\partial_{jR}\Phi_2 - F_{iL}x_{iL}\Phi_2 + F_{iR}x_{iR}\Phi_2\,.$$

Expanding both Φ_2 and F in left and right spherical harmonics as

$$\Phi_2 = \sum_{klmn} c_{klmn} Y^L_{km} Y^R_{ln} \quad \text{and} \quad F_{ia}x_{ia} = \sum_{klmn} f^a_{klmn} Y^L_{km} Y^R_{ln}$$

we get for fixed $klmn$, setting $c = c_{klmn}$, $f^a = f^a_{klmn}$ and $p = \frac{1}{2} + l(l+1) + k(k+1)$ the following expression

$$pc^2 - cf^L + cf^R = p(c - \frac{1}{2p}f^L + \frac{1}{2p}f^R)^2 - \frac{1}{4p}(f^L - f^R)^2\,.$$

Integrating out the c's and putting everything back this leaves us with the additional term

$$-(F_{iL}x_{iL} - F_{iR}x_{iR})\frac{1}{4(\frac{1}{2} - \partial_\mu\partial_\mu)}(F_{iL}x_{iL} - F_{iR}x_{iR})$$

in the action (4.15).

D. Representation of the SO(6)-intertwiners and Clifford algebra

We will use the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which satisfy

$$\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k\,. \tag{D.1}$$

With these we define the 4-dimensional antisymmetric matrices

$$\begin{aligned} &\gamma^1_L = \sigma^1 \otimes \sigma^2\,, \qquad \gamma^2_L = \sigma^2 \otimes 1\,, \qquad \gamma^3_L = \sigma^3 \otimes \sigma^2\,, \\ &\gamma^1_R = i\,\sigma^2 \otimes \sigma^1\,, \qquad \gamma^2_R = i\,1 \otimes \sigma^2\,, \qquad \gamma^3_R = i\,\sigma^2 \otimes \sigma^3\,. \end{aligned} \tag{D.2}$$

They are the intertwiners between SU(4) $\otimes$ SU(4) and SO(6) and fulfill the following relations:

$$
\begin{aligned}
(\gamma_L^i)^\dagger &= \gamma_L^i\,,\\
(\gamma_R^i)^\dagger &= -\gamma_R^i
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma_L^i\gamma_L^j &= \delta^{ij} + i\epsilon^{ij}_k\gamma_L^k\,,\\
\gamma_R^i\gamma_R^j &= -\delta^{ij} - \epsilon^{ij}_k\gamma_R^k\,,\\
[\gamma_L^i,\gamma_R^j] &= 0\,.
\end{aligned}
$$

We can now define the 8-dimensional representation of the SO(6)-Clifford algebra as

$$\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^{\mu\dagger} & 0 \end{pmatrix}, \tag{D.3}$$

with the desired anticommutation relations

$$\{\Gamma^\mu,\Gamma^\nu\} = \begin{pmatrix} \gamma^\mu\gamma^{\nu\dagger}+\gamma^\nu\gamma^{\mu\dagger} & 0 \\ 0 & \gamma^{\mu\dagger}\gamma^\nu+\gamma^{\nu\dagger}\gamma^\mu \end{pmatrix} = 2\delta^{\mu\nu}\,.$$

The chirality operator in this basis is

$$\Gamma = i\Gamma_L^1\Gamma_L^2\Gamma_L^3\Gamma_R^1\Gamma_R^2\Gamma_R^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The 8-dimensional SO(6)-rotations are generated by

$$\Sigma_8^{\mu\nu} = -\frac{i}{4}[\Gamma^\mu,\Gamma^\nu] = -\frac{i}{4}\begin{pmatrix} \gamma^\mu\gamma^{\nu\dagger}-\gamma^\nu\gamma^{\mu\dagger} & 0 \\ 0 & \gamma^{\mu\dagger}\gamma^\nu-\gamma^{\nu\dagger}\gamma^\mu \end{pmatrix}.$$

E. The Dirac operator in spherical coordinates

For a general riemannian manifold with metric

$$g = g_{\mu\nu}dx^\mu dx^\nu$$

the Christoffel symbols are given by

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})\,. \tag{E.1}$$

We can change to a non-coordinate basis (labeled by latin indices in contrast to the greek indices for the coordinates) by introducing the vielbeins e_a^μ with

$$
\begin{aligned}
e^a_\mu e^\mu_b &= \delta^a_b\,,\\
g_{\mu\nu} &= e^a_\mu e^b_\nu\delta_{ab}\,, \qquad g^{\mu\nu} = e^\mu_a e^\nu_b\delta^{ab}\,.
\end{aligned}
$$

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With these, the Dirac operator is given by

$$D = -i\gamma^a e^\mu_a (\partial_\mu + \frac{1}{4}\omega_{\mu ab}[\gamma^a, \gamma^b])\,,$$

where the γ^a form a flat Clifford algebra, i. e.

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab}\,, \qquad \gamma^{a\dagger} = \gamma^a$$

and the spin connection ω fulfills

$$\partial_\mu e^a_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda + \omega_\mu{}^a{}_b\, e^b_\nu = 0\,. \tag{E.2}$$

E.1 The Dirac operator on $\mathbb{R}^6$ in spherical coordinates

We will now write down the flat SO(6) Dirac operator D_6 by splitting $\mathbb{R}^6$ into $\mathbb{R}^3_L \times \mathbb{R}^3_R$ and introducing spherical coordinates on both the left and right hand side. The flat metric becomes

$$\begin{aligned} g_6 = {} & r_L^2\, d\theta_L \otimes d\theta_L + r_L^2 \sin^2\theta_L\, d\phi_L \otimes d\phi_L + dr_L \otimes dr_L + \\ & + r_R^2\, d\theta_R \otimes d\theta_R + r_R^2 \sin^2\theta_R\, d\phi_R \otimes d\phi_R + dr_R \otimes dr_R\,. \end{aligned} \tag{E.3}$$

Looking at the formula for the Christoffel symbols (E.1), we see that all the symbols with both right and left indices vanish. For the symbols with only right or only left indices we get

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta\,, \tag{E.4}$$

$$\Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta} = \Gamma^\phi_{\phi\theta}\,, \tag{E.5}$$

$$\Gamma^r_{\theta\theta} = -r\,, \tag{E.6}$$

$$\Gamma^r_{\phi\phi} = -r\sin^2\theta\,, \tag{E.7}$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r} = \Gamma^\theta_{\theta r}\,, \tag{E.8}$$

$$\Gamma^\phi_{r\phi} = \frac{1}{r} = \Gamma^\phi_{\phi r}\,, \tag{E.9}$$

where we have dropped the left or right subscript for simplicity. All other symbols vanish. We want to go to a non-coordinate basis by introducing the vielbeins

$$e^{1_L}_{\theta_L} = r_L\,; \qquad e^{2_L}_{\phi_L} = r_L \sin\theta_L\,; \qquad e^{3_L}_{r_L} = 1\,; \tag{E.10}$$

$$e^{1_R}_{\theta_R} = r_L\,; \qquad e^{2_R}_{\phi_R} = r_R \sin\theta_R\,; \qquad e^{3_R}_{r_R} = 1\,. \tag{E.11}$$

Calculating the spinor connection by (E.2), we again see that all the terms with both left and right indices vanish. The terms with only left or only right indices are

$$\omega_\phi{}^1{}_2 = -\cos\theta = -\omega_\phi{}^2{}_1\,, \tag{E.12}$$

$$\omega_\phi{}^2{}_3 = \sin\theta = -\omega_\phi{}^3{}_2\,, \tag{E.13}$$

$$\omega_\theta{}^1{}_3 = 1 = -\omega_\theta{}^3{}_1\,, \tag{E.14}$$

where we again dropped the left or right subscripts. Putting all this together we see that D_6 splits up into a left part D_{3L} and a right part D_{3R} as

$$D_6 = D_{3L} + D_{3R} \tag{E.15}$$

with

$$\begin{aligned} D_{3L} &= -i\overline{\Gamma}^1_L \frac{1}{r_L}\left(\partial_{\theta_L} + \frac{\cos\theta_L}{\sin\theta_L}\right) - i\overline{\Gamma}^2_L \frac{1}{r_L \sin\theta_L}\partial_{\phi_L} - i\overline{\Gamma}^3_L\left(\partial_{r_L} + \frac{1}{r_L}\right), && \text{(E.16)} \\ D_{3R} &= -i\overline{\Gamma}^1_R \frac{1}{r_R}\left(\partial_{\theta_R} + \frac{\cos\theta_R}{\sin\theta_R}\right) - i\overline{\Gamma}^2_R \frac{1}{r_R \sin\theta_R}\partial_{\phi_R} - i\overline{\Gamma}^3_R\left(\partial_{r_R} + \frac{1}{r_R}\right). && \text{(E.17)} \end{aligned}$$

where the $\overline{\Gamma}$ have to form a SO(6) Clifford algebra.

E.2 The Dirac operator on $S^2 \times S^2$

We now want to calculate the curved Dirac operator D_4 on $S^2 \times S^2$ in the spherical coordinates of the spheres (they are the same spherical coordinates we used before, now restricted to the spheres). The metric on $S^2 \times S^2$ with radii r_L and r_R is

$$\begin{aligned} g_4 = {} & r_L^2\, d\theta_L \otimes d\theta_L + r_L^2 \sin^2\theta_L\, d\phi_L \otimes d\phi_L + \\ & + r_R^2\, d\theta_R \otimes d\theta_R + r_R^2 \sin^2\theta_R\, d\phi_R \otimes d\phi_R\,. \end{aligned}$$

The metric is the same as (E.3) restricted to the spheres, so the Christoffel symbols are the same as (E.4) and (E.5). Again introducing the vielbeins

$$\begin{aligned} e^{1_L}_{\theta_L} &= r_L\,; \qquad & e^{2_L}_{\phi_L} &= r_L \sin\theta_L\,; && \text{(E.18)} \\ e^{1_R}_{\theta_R} &= r_L\,; \qquad & e^{2_R}_{\phi_R} &= r_R \sin\theta_R\,, && \text{(E.19)} \end{aligned}$$

we see that also the spin connection is the same as (E.12), and therefore we can again split D_4 into a right part D_{2R} and a left part D_{2L} as $D_4 = D_{2L} + D_{2R}$ with

$$\begin{aligned} D_{2L} &= -i\widetilde{\Gamma}^1_L \frac{1}{r_L}\left(\partial_{\theta_L} + \frac{\cos\theta_L}{\sin\theta_L}\right) - i\widetilde{\Gamma}^2_L \frac{1}{r_L \sin\theta_L}\partial_{\phi_L}\,, && \text{(E.20)} \\ D_{2R} &= -i\widetilde{\Gamma}^1_R \frac{1}{r_R}\left(\partial_{\theta_R} + \frac{\cos\theta_R}{\sin\theta_R}\right) - i\widetilde{\Gamma}^2_R \frac{1}{r_R \sin\theta_R}\partial_{\phi_R}\,, && \text{(E.21)} \end{aligned}$$

where the $\widetilde{\Gamma}$ form a flat SO(4) Clifford algebra.

E.3 SO(3) × SO(3)-covariant form of the Dirac operator on $S^2 \times S^2$

The flat SO(6) Dirac operator D_6 was split into a left part D_{3L} and a right part D_{3R} using spherical coordinates in (E.15). Of course, D_6 can also be written in the usual euclidean coordinates as

$$D_6 = -i\Gamma^\mu \partial_\mu\,,$$

where again we can split it into a left and a right part as

$$D_6 = D_{3L} + D_{3R}$$

with

$$
\begin{aligned}
D_{3L} &= -i\Gamma^i_L \partial_i\,, \qquad D_{3R} = -i\Gamma^i_R \partial_i\,, && \text{(E.22)}\\
\{D_{3L}, D_{3R}\} &= 0. && \text{(E.23)}
\end{aligned}
$$

We have left open which representation of the SO(6) Clifford algebra we want to use for the $\overline{\Gamma}$ in (E.16), (E.17), but Γ in (E.22) is really the representation given by (4.9). We will now relate the two expressions for the Clifford algebra and the Dirac operator by first defining

$$
J_{iL} = -i\epsilon_{ijk} x_{jL} \partial_{kL} \qquad \text{and} \qquad J_{iR} = -i\epsilon_{ijk} x_{jR} \partial_{kR}
$$

and noting that

$$
\left(\frac{\Gamma^i_L x_{iL}}{r_L}\right)^2 = \left(\frac{\Gamma^i_R x_{iR}}{r_R}\right)^2 = 1\,. \tag{E.24}
$$

We calculate that

$$
\left(\frac{\Gamma^j_L x_{jL}}{r_L}\right)^2 \Gamma^i_L \partial_{iL} = \left(\frac{\Gamma^j_L x_{jL}}{r_L}\right) \left(\frac{x_{iL}\partial_{iL}}{r_L} - \frac{1}{r_L}\begin{pmatrix} \gamma^i_L & 0 \\ 0 & \gamma^i_L \end{pmatrix} J_{iL}\right), \tag{E.25}
$$

$$
\left(\frac{\Gamma^j_R x_{jR}}{r_R}\right)^2 \Gamma^i_R \partial_{iR} = \left(\frac{\Gamma^j_R x_{jR}}{r_R}\right) \left(\frac{x_{iR}\partial_{iR}}{r_R} + \frac{i}{r_R}\begin{pmatrix} \gamma^i_R & 0 \\ 0 & \gamma^i_R \end{pmatrix} J_{iR}\right), \tag{E.26}
$$

and therefore

$$
D_{3L} = -i\left(\frac{\Gamma^j_L x_{jL}}{r_L}\right)\left(\partial_{r_L} - \frac{1}{r_L}\begin{pmatrix} \gamma^i_L & 0 \\ 0 & \gamma^i_L \end{pmatrix} J_{iL}\right), \tag{E.27}
$$

$$
D_{3R} = -i\left(\frac{\Gamma^j_R x_{jR}}{r_R}\right)\left(\partial_{r_R} + \frac{i}{r_R}\begin{pmatrix} \gamma^i_R & 0 \\ 0 & \gamma^i_R \end{pmatrix} J_{iR}\right). \tag{E.28}
$$

Comparing this with (E.16), (E.17) we see that

$$
\overline{\Gamma}^3_L = \left(\frac{\Gamma^i_L x_{iL}}{r_L}\right) \qquad \text{and} \qquad \overline{\Gamma}^3_R = \left(\frac{\Gamma^i_R x_{iR}}{r_R}\right), \tag{E.29}
$$

as the J_L and J_R have no radial components. From (E.27), (E.28) we can also deduce that

$$
\left[\overline{\Gamma}^i_L, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = 0 = \left[\overline{\Gamma}^i_R, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \tag{E.30}
$$

and

$$
\left\{\overline{\Gamma}^i_R, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\} = 0 = \left\{\overline{\Gamma}^i_L, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}. \tag{E.31}
$$

The curved Dirac operator D_4 on $S^2 \times S^2$ expressed in the spherical coordinates of the spheres also splits up as $D_4 = D_{2L} + D_{2R}$ with right part D_{2R} and left part D_{2L} given in (E.20),(E.21). Comparing this with (E.16), (E.17), we see that the dependence on the tangential coordinates is the same in both expressions. With (E.30), (E.31) we see that the matrices $-i\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\overline{\Gamma}^3_L\overline{\Gamma}^i_L$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\overline{\Gamma}^3_R\overline{\Gamma}^j_R$ for $i,j = 1,2$ form a SO(4) Clifford algebra and can therefore be used as the $\widetilde{\Gamma}$. Note that this representation is still reducible, a problem we deal with in section 8.1.1. Now we can get a simple relation between the D_3 restricted on the spheres and the D_2

$$-\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\left(i\overline{\Gamma}^3_L D_{3L}|_{res.} - \frac{1}{r_L}\right) = D_{2L}\,,$$
$$-i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left(i\overline{\Gamma}^3_R D_{3R}|_{res.} - \frac{1}{r_R}\right) = D_{2R}\,.$$

Inserting (E.27), (E.28) and using (E.29) together with (E.24) we find that

$$D_{2L} = \frac{1}{r_L}\left(\Gamma^i_L J_{iL} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \tag{E.32}$$

$$D_{2R} = \frac{1}{r_R}\left(\Gamma^i_R J_{iR} + i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right). \tag{E.33}$$

Setting $r_L = r_R = 1$ for simplicity, the Dirac operator D_4 on $S^2 \times S^2$ takes the form (8.1).

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